6-20-2014

Investigating Children's Intuitive And Analytical Thinking About Path Length As A Developmental Phenomenon

Cheryl Lynn Eames
Illinois State University, cleames@ilstu.edu

INVESTIGATING CHILDREN'S INTUITIVE AND ANALYTICAL THINKING ABOUT PATH LENGTH AS A DEVELOPMENTAL PHENOMENON

Cheryl L. Eames

280 Pages December 2014

Children's conceptions of length measurement has been the focus of research that has built on the work of Piaget and his colleagues to produce developmental accounts for the acquisition of conceptual and procedural knowledge. Prior research focused on children's developing conceptions of length measurement for straight or rectilinear paths; however, little is known about how these conceptions grow beyond the elementary grades. The present study increased the scope of this research beyond elementary grades to include middle and secondary level students, exploring the development of students' intuitive and analytical thinking for determining the length of a curved path across a wide span of development. Finally, this study extends a hypothetical learning trajectory (LT), to include intuitions for path length.

I administered a written LT-based length assessment to 82 students in Grades 4, 6, 8, and 10, which I coded using a length LT. Based on this assessment, I selected four participants from each of Grades 4, 6, 8, and 10 as representatives of four levels of the LT. I conducted two individual task-based interviews (Goldin, 2000) with each of the 16

participants, which I analyzed using codes from research on path length intuition (Chiu, 1996) and emergent codes generated through a constant comparative method. I then tracked the frequency of each code to explore developmental patterns.

Results suggest that the tasks included in this study effectively differentiated students' thinking at different LT levels. These findings are consistent with Fischbein's theory of intuition (1987), which describes intuition as a developmental phenomenon. Participants who exhibited different levels of sophistication, measured by the length LT, exhibited different ways of evoking intuitions in terms of (a) intuitions and analytical strategies overall, (b) each individual intuition, and (c) analytical strategies with embedded intuitions. Furthermore, findings confirm conjectured concepts and processes outlined in the LT.

INVESTIGATING CHILDREN'S INTUITIVE AND ANALYTICAL THINKING ABOUT PATH LENGTH AS A DEVELOPMENTAL PHENOMENON

CHERYL L. EAMES

A Dissertation Submitted in Partial
Fulfillment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

ILLINOIS STATE UNIVERSITY

2014

INVESTIGATING CHILDREN'S INTUITIVE AND ANALYTICAL THINKING ABOUT PATH LENGTH AS A DEVELOPMENTAL PHENOMENON

CHERYL L. EAMES

COMMITTEE MEMBERS:

Jeffrey E. Barrett, Co-chair

Craig J. Cullen, Co-chair

David Klanderman

Richard Lesh

Jennifer Tobias

ACKNOWLEDGMENTS

I would like to thank my committee members for the countless hours they spent supporting me in designing and carrying out this dissertation study. First, I would like to thank my co-chairs, Jeffrey Barrett and Craig Cullen, for inviting me to participate in their research as a collaborator. My work has been inspired by the four years I spent as a member of their research team. Next, I would like to thank Jennifer Tobias for her attention to detail and David Klanderman for lending his expertise on children's thinking and learning about length measurement and thoughtful feedback to my work. I would like to thank Richard Lesh for asking me difficult questions, facilitating new collaborative relationships, and encouraging me to open my mind to new theoretical perspectives and research methods.

I would also like to thank all of the other members of the Children's Measurement Project Research Team: Amanda Miller, Melike Kara, Pam Beck, Doug Clements, Julie Sarama, and Doug Van Dine. It has been a privilege to call you my colleagues and my friends. Finally, I would first like to thank my family for all of their support and encouragement as I worked on this dissertation study. I would especially like to thank Scott, Maia, and Ava Eames who put up with my many hours of typing away. This dissertation is dedicated to you.

C. L. E.

CONTENTS

TABLES

FIGURES

CHAPTER 1

INTRODUCTION AND RATIONALE

Introduction

Early research on children's conceptions of length measurement conducted by Piaget and his colleagues focused on the development of the logical operations of conservation and transitivity and the development of an iterable unit of length through subdividing and ordering those subdivisions (Piaget, Inhelder, & Szeminksa, 1960). Mathematics educators later extended this work to investigate the development of children's capabilities for concepts and procedures related to unit: unit iteration, tiling or structuring with units, relationships among different units, additivity, and understanding of the zero point (Lehrer, 2003). However, most of the research on children's conceptions of length measurement has been done within the Piagetian tradition of coupling the study of measurement with the study of space (e.g., Barrett & Clements, 2003; Barrett et al., 2006; Barrett, et al, 2012; Battista, 2006; Clarke, Cheeseman, McDonough, & Clarke, 2003; Hiebert, 1981; Sarama & Clements, 2009; Steffe & Hirstein, 1976). This work has focused mainly on elementary children's capabilities of measuring lengths of rectilinear paths in one- and two-dimensional space (see Figures 1, 2, and 3) and has produced developmental accounts for the acquisition of sophistication in conceptual and procedural knowledge for length measurement (Battista, 2006; Clarke, Cheeseman, McDonough, & Clarke, 2003; Piaget, Inhelder, & Szeminksa, 1960; Sarama & Clements, 2009).

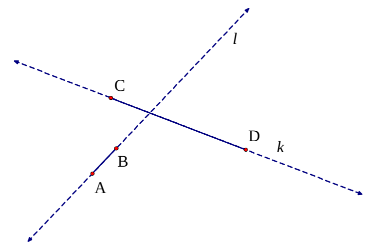


Figure 1. Comparing lengths of segments that are not on the same line (Battista, 2006; Piaget, Inhelder, & Szeminksa, 1960).

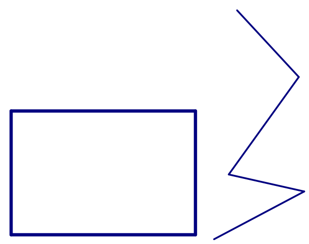


Figure 2. Finding lengths of bent paths and perimeter (Battista, 2006; Piaget, Inhelder, & Szeminksa, 1960).

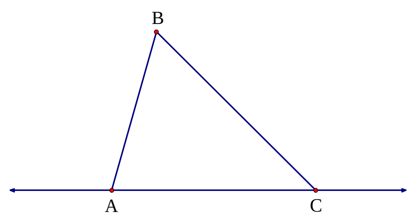


Figure 3. Triangle inequality (Barrett et al., 2006) or shortest distance between points.

Although a strong basis of empirical evidence exists for the developmental accounts of *young* children's conceptions of length measurement, little is known about how these concepts continue to grow beyond the elementary grades to become more sophisticated and coherent. Researchers have called for the elaboration of these developmental accounts for measurement to middle school students (Daro, Mosher, Corcoran, 2011). Moreover, a growing number of researchers are calling for this work to further extend the research that was inspired by Piaget and his colleagues (i.e. Battista, 2006; Sarama & Clements, 2009) by including the investigation of children's conceptual and procedural knowledge related to linear measurement in the context of curved paths (Clements et al., in press) as well as intuitions for path length (Chiu, 1996). This study seeks to extend the work of Piaget in this manner and to provide an empirical basis for expanding developmental accounts into the Middle and High School Levels.

Extending the Work of Piaget

Osborne (1976) outlined four problems of length and distance to consider in the teaching and learning of measurement: (a) comparing lengths of segments on two different lines, (b) measuring lengths of bent paths, (c) finding the shortest distance between two points, and (d) determining the length of a curve. Piaget and many researchers who extended his work have used tasks of the first three types described by Osborne to inform the articulation of developmental accounts of elementary children's conceptions of length measurement (see Figures 1, 2, and 3).

Osborne (1976) claimed that determining the length of a curve "is a step beyond school mathematics" because "the solution depends on limit processes, the additivity property, extended to allow for adding an infinite number of segments" (p. 24). However, a small body of research (see Clements et al, in press; Grugnetti, Rizza, & Marchini, 2007) suggests that, before students have access to calculus as a conceptual tool, determining the length of a curve is a task that has potential instructional value for addressing measurement from a mathematical perspective (Osborne, 1976) using informal limit arguments, an approach that has been recommended in the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010).

Before instruction in calculus, the length of a curve could be determined by measuring it directly using a string or by a discrete linear approximation. A discrete linear approximation of a continuous curve involves representing the curve as a collection of linear segments (Figure 4); the length of the curve can be approximated by adding the lengths of the linear segments.

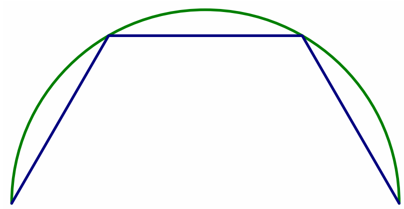


Figure 4. Representing a curve as a collection of linear segments

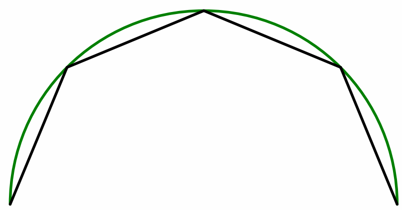


Figure 5. Increasing the number and decreasing the length of each segment to reduce error.

This representation highlights key ideas about the nature of measure: all measurement of continuous quantity is an approximation, increasing the number and decreasing the length of the segments provides a better representation of the curve and reduces approximation error (Figure 5), an approximation can be an overestimate (Figure 6) or an underestimate (Figure 5), and approximation error can be reduced by averaging over- and underestimates for the length of a curve (Figure 7).

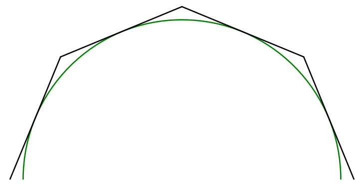


Figure 6. Overestimating the length of a curve.

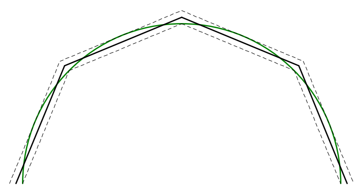


Figure 7. Averaging over- and underestimates using Archimedes' method (see Traub, 1984).

The task of determining the length of a curve by approximation provides the potential opportunity to investigate children's thinking about concepts related to unit, such as their developing capabilities for coping with units of units (both sub and superordinate units) with efficiency and precision, and coordination of other features with linear measures, such as curvature (i.e., using smaller units to measure a tighter curve will result in a more precise measure).

Researchers have mainly carried on the Piagetian tradition of investigating children's developing conceptual and procedural knowledge for length measurement (i.e., Battista, 2006; Barrett & Clements, 2003; Clements et al, in press). However, researchers in psychology and mathematics education have shown that children, as well as adults, also possess intuitive knowledge for path length. For example, several studies by mathematics educators (see Barrett & Clements, 2003; Chiu, 1996; Clements, Battista, Sarama, & Swaminathan, 1996) and psychologists (e.g., Thordyke, 1981; Kosslyn, Pick & Fariello, 1974; Luria, Kinney; & Weissman, 1967; Pressey, 1974) have documented the prevalence of the complexity intuition (Chiu, 1996), which is characterized by an attention to the number of segments or turns when comparing rectilinear paths. In their work, Barrett and Clements (2003) suggested that, at the elementary level, children's developing abstractions for linear measurement, with respect to establishing exact correspondence between counting and linear dimensions of paths, interacts with intuitive thinking for path length. However, no prior study has examined how intuitive thinking for path length changes or how intuitive thinking for path length interacts with conceptual and procedural knowledge for length measurement across a large span of development.

Purpose of the Study

The purpose of this study is to extend the literature on children's conceptions of length measurement in two-dimensional space in three important ways. First, this study increases the scope of the body of research on students' thinking in the context of length measurement beyond elementary aged children to include middle and secondary level students. Next, this study involves the exploration of the development of students' thinking in the context of approximating the length of a curve (see Figures 4-8) across a wide span of development, which has not been addressed in prior studies (see Figures 1, 2, and 3). Finally, this study seeks to extend existing developmental accounts for the learning of length measurement to explore interactions among students' conceptual and procedural knowledge for length measurement with their intuitions for rectilinear and curvilinear path length.

Research Questions

This study seeks to explore elementary, middle, and secondary school level students' intuitive and analytical thinking when comparing rectilinear and curvilinear paths in two-dimensional space by length. Specifically, this study examines the intuitions and analytical strategies that students at different levels of sophistication for length measurement use for path length. The following questions guided task design and subsequent analysis:

1. What intuitions and analytical strategies do students use when comparing sets of rectilinear or curvilinear paths by length?
2. How does their use of intuitive and analytical thinking for path length change or develop across levels of sophistication for length measurement?

CHAPTER II

THEORETICAL GROUNDING AND REVIEW OF RELATED LITERATURE

Theoretical Framework

To explore students' intuitions and analytical strategies for path length and how their use of intuitive or analytical thinking for path length changes or develops across levels of sophistication for length measurement, I required a theoretical tool that could serve two main purposes. First, I needed a theory that would allow me to differentiate intuitive thinking from analytical thinking. Second, I needed a theoretical vantage point from which to identify students at different levels of sophistication for length measurement for the purpose of selecting a sample of students at the same and adjacent levels. A single theoretical framework could not meet both of these criteria. However, a synthesis of key features of Fischbein's (1987) theory on intuition and a hypothetical learning trajectory (LT) for length measurement (Clements et al., in press) provided a theoretical framing that could serve both purposes. I begin this chapter with sections in which I describe components of these frameworks, which are most germane to the present study and conclude with a review of the related literature that was informed by the synthesis of these two theoretical perspectives.

Fischbein's Theory on Intuition

The most relevant aspect of Fischbein's (1987) theory on intuition for the present study is an operational definition for intuition that allows for distinguishing intuitive thinking from analytical thinking or perception. Fischbein defined an intuition as "a primary phenomenon which may be described but which is not reducible to more elementary components" (p. ix). He characterized an intuition as having the appearance of being a self-evident and self-consistent cognition, much like perceiving a color or experiencing an emotion. Fischbein argued that human beings possess a natural and almost instinctual belief in the existence of some absolute certitude, which has manifested itself throughout the history of mathematics and science. It is this need for certitude, "our fundamental need 'to see' with our mind, as we see with our eyes" (p. 7), that motivates intuitive thought. The sections below elaborate Fischbein's definition of intuition by outlining the properties and classes of intuitions.

Properties of intuitions. Fischbein (1987) described intuitive knowledge as a self-evident, immediate cognition. For example, in the case of path length, one intuitively knows that the shortest distance between two points is a straight line (Chiu, 1996; Fischbein, 1987). This statement is "accepted as being immediate and self-evident without feeling the need for a proof either formal or empirical" (Fischbein, 1987, p. 13). Fischbein argued that this self-evident nature leads to three other key properties of intuitions: extrapolativeness, coerciveness, and globality.

Fischbein (1987) argued that an intuition always exceeds observable facts. An intuition, then, is a theory; "it implies an extrapolation beyond the directly accessible information" (p. 13). For example, one does not need intuition to *see* that pairs of opposite angles of two intersecting lines are congruent. However, one uses intuition to accept the universality of this property.

Although intuitions appear to be self-evident and even autonomous, Fischbein (1987) noted that they are also robust and are deeply rooted in one's mental organization.

That is, intuitions are coercive (Fischbein, 1987). They “appear, generally, as absolute, unchangeable ones” (p. 14). Altering, eliminating, or controlling an intuition would require “a profound, structural transformation in large areas of mental activity” (p. xi). Therefore, according to Fischbein, the coersive nature of intuitions contributes to the perpetuation of wrong interpretations. For example, when comparing paths by length, both children and adults have a propensity to attend to the complexity, such as the number of segments or turns in a path, rather than overall length (e.g., Barrett & Clements, 2003; Chiu, 1996; Kosslyn, Pick, & Fariello, 1974; Luria, Kinney, & Weissman, 1967; Pressey, 1974; Thorndyke, 1981).

Fischbein (1987) argued that the globality of an intuition is a consequence of its self-evident nature. “A certain statement accepted as self-evident is also accepted globally as a structured, meaningful, unitary representation” (p. 14). An intuition, a global and synthetic view, is a direct and quick view without preliminary analysis. Furthermore, the globality of intuition is revealed by a repeated application of an intuition, informed by a recognition that one context is analogous to another. For example, the global character of the complexity intuition described above is evinced by both an immediate application without preliminary analysis as well as its application across multiple tasks and contexts.

This global property of intuitions serves to distinguish between intuitive and analytical thinking. Whereas intuitive thinking is direct and quick without preliminary analysis, analytical thinking proceeds in a step-by-step manner, in which one notion is connected to the next. For the present study, a student’s response to a task is regarded as intuitive thinking only if it appears to be an immediate, direct, and global solution. A response that appears to exhibit step-by-step reasoning is regarded as analytical thinking.

Classes of intuitions. To further elaborate on his definition of intuition, Fischbein (1987) offered two approaches for classifying intuitions, one based on roles or origins. The classification system that is based on roles also considers the relationship between an intuition and the solution to a particular problem. In this system, intuitions can be affirmatory, conjectural, anticipatory, or conclusive. In the case of an affirmatory intuition, one affirms or makes a claim. A conjectural intuition is one in which an assumption about future events is expressed. Anticipatory and conclusive intuitions represent phases in the process of solving a problem. Anticipatory intuitions express a preliminary, global view that precedes an analytical solution to a problem. Conclusive intuitions summarize in a global, structured vision the solution to a problem that had previously been elaborated. For the present study, because students' claims about comparisons among paths by length are being observed, the intuitions subject to examination are affirmatory.

Fischbein's (1987) alternative system for classifying intuitions is based on the origin of an intuition and distinguishes intuitions as either primary or secondary. Primary intuitions are "those cognitive beliefs which develop in individuals independently of any systematic instruction as an effect of their personal experience" (p. 64). Secondary intuitions, however, are not produced by natural, normal experiences. Secondary intuitions are formed when a learned conception is transformed into a belief. For example, the claim that the sum of the interior angles of a triangle is 180 degrees, regardless of its shape, is not self-evident. It can be proved. Fischbein explained that if one comes to "see *directly* that the sum *must necessarily* remain constant (because of inner compensation)" (p. 68) one has acquired a new secondary intuition.

Intuition as a developmental phenomenon. The term "primary intuition" does not indicate that an intuition is innate (Fischbein, 1987). Both primary and secondary intuitions are learned, and they are both "always the product of an ample and lasting practice in some field of activity" (p. 69). Therefore, Fischbein argued, "intuitions are a developmental phenomena, and their structure changes as an effect of experience and a general intellectual development" (p. 115). A child's intuition use changes over time. For example, classic Piagetian conservation of quantity tasks have been used to show that young children intuitively apprehend the number of discrete objects laid in a row based on the length of the row rather than a count of the objects (see Piaget and Szeminska, 1964, p. 99). This apprehension is "intuitive, global, without hesitation, based on configurations rather than on operational criteria" (Fischbein, 1987, p. 65). This apprehension is a primary intuition.

Over time, new intuitions develop "based on the composability and reversibility of intellectual operations: intuitions related to conservation capacities, to the notions of number and cardinality, to elementary logical and arithmetical operations" (Fischbein, 1987, p. 65). For example, on Piagetian conservation of quantity tasks, over time children begin to attend to the number of discrete objects laid in a row rather the length of the row. Although new intellectual operations become available to the child, "the reactions of the child remain, nevertheless, global, direct, and his interpretations appear to him as self-evident" (p. 65). These new intellectual operations become the essential texture of intuitive reactions. That is, a child's response to a task may be based on these intellectual operations, the response may still “display the properties of an intuitive cognition; it appears subjectively non-explicitly justified and *a priori* evident” (p. 65).

Fischbein’s (1987) theory on intuition provides an operational definition that allows for distinguishing intuitive thinking from analytical thinking; however, it does not provide a structure for describing the hierarchic development of children’s conceptions for length measurement. A hypothetical learning trajectory (LT) for length measurement (Clements et al, in press) addresses this aspect of the present study by providing a means for describing how children’s thinking about rectilinear and curvilinear path length changes or develops across levels of sophistication for length measurement. An LT has three parts: (a) an instructional goal, (b) a likely path for learning through increasingly sophisticated levels of thinking, and (c) the instructional tasks that engender the mental processes or actions that support children’s growth through those levels (Clements & Sarama, 2007). In the present study, the LT for length measurement serves as a tool for describing and differentiating children’s responses according to those levels of sophistication.

Hierarchic Interactionalism

LTs are a central feature of hierarchic interactionalism (HI), which is a theory of cognitive development that is represents a synthesis of empiricism, (neo)nativism, and interactionalism (Clements & Sarama, 2007). “Hierarchic” indicates the influence and interaction of domain-general and domain-specific cognitive components and the interactions of innate competencies, internal resources, and experience. LTs originate from a key tenet of HI, which postulates that children progress through domain-specific levels of understanding in ways that can be characterized by specific mental objects and actions (i.e., both concept and process) that build hierarchically on previous levels (Clements & Sarama, 2007).

Clements and Sarama (2007) elaborated HI using 12 tenets. The first of the twelve tenets addresses *developmental progressions*. The next five tenets of HI, *domain specific progression*, *hierarchic development, cyclic concretization, co-mutual development of concepts and skills*, and *progressive hierarchization*, address the levels of and a child's movement within a developmental progression. Three of the twelve tenets of HI, *initial bootstraps*, *different developmental courses*, and *environment and culture*, explain how these developmental progressions are guided. Two of the twelve tenets, *consistency of developmental progressions and instruction* and *LTs*, both address effective instruction and developmental progressions. The final tenet of HI, *instantiation of LTs*, addresses some of the limitations and affordances of LTs.

Of these 12 tenets of HI, five address key assumptions of HI that are relevant to the investigation of intuitive and analytical thinking about path length as a developmental phenomenon. In the following sections I describe these five relevant tenets as well as how they contribute to the framing of the present study

Developmental progressions. According to the perspective of HI, "knowledge is acquired along developmental progressions of thinking" (Clements & Sarama, 2007, p. 464). These developmental progressions are "consistent with children's intuitive knowledge and patterns of thinking and learning at various levels of development" (Clements & Sarama, 2007, p. 464). Hence, each level of development is characterized by different concepts and processes. Therefore, based on this tenet, a key assumption of the present study is that children who are at different levels within a developmental progression possess different concepts and processes, so they would exhibit different intuitions and analytical strategies for rectilinear and curvilinear path length.

Domain-specific progression. Clements and Sarama (2007) emphasized that developmental progressions address specific mathematical topics; therefore, developmental progressions must be domain-specific. Knowledge is the "main determinant of the thinking within each progression, although hierarchic interactions occur at multiple levels within and between topics, as well as general cognitive processes" (Clements & Sarama, 2007, p. 464). From the perspective that intuition is a cognition (Fischbein, 1987), this tenet of HI supposes that a hierarchic interaction exists within and between knowledge for length measurement and intuition for path length.

Hierarchic development. Development is an "interactive interplay among specific components of knowledge and processes" (Clements & Sarama, 2007, p. 464). Each level of a developmental progression builds hierarchically out of the concepts and processes that constitute the previous levels. These levels are organized according to increasing "sophistication, complexity, abstraction, power, and generality" (Clements & Sarama, 2007, p. 465).

The process of learning or development is incremental and gradual. Various types of thinking develop in tandem, "but a critical mass of ideas from each level must be constructed before thinking characteristic of the subsequent level becomes ascendant in the child's thinking and behavior" (Clements and Sarama, 2007, p. 465). As the child moves through developmental progressions, previous levels of thinking are not deleted from memory. These levels of thinking become more explicit mental representations, which do not erase the earlier representations. In fact, these early representations emerge as fallback strategies under conditions of increased stress, when confronted with more complex tasks, or when another process fails (Clements & Sarama, 2007).

This tenet of HI indicates that, although a child may be operating predominantly at a particular level for length measurement, he or she may exhibit evidence of higher or lower levels of thinking. Therefore, within a group of children who are appear to be operating at the same length LT level based on their responses to a collection of tasks, individual differences may be observed as children cope with more complex tasks. Furthermore, this tenet of HI supposes that children, who are operating predominantly at the same LT level, may exhibit some individual differences in aspects of length measurement outside the current LT, such as intuitions and analytical strategies for rectilinear and curvilinear paths.

Co-mutual development of concepts and skills. Concepts and skills develop in constant interaction; concepts and skills encompass symbolic representations, utilization competence, and general cognitive skills (Clements & Sarama, 2007). As a child ascends through a developmental progression, he or she gradually makes "connections between various mathematically relevant concepts and procedures, weaving ever more robust understandings that are hierarchical" (p. 465). Therefore, the domain-specific developmental progression for length measurement that is reflected in the length LT (Clements et al., in press) outlines levels of increasingly sophisticated conceptual and procedural knowledge for key length measurement concepts.

Learning trajectories. A fruitful approach for instruction is based on LTs (Clements & Sarama, 2007). "On the basis of the hypothesized specific mental constructions (mental actions-on-objects) and patterns of thinking that constitute children's thinking, curriculum developers design instructional tasks that include external objects and actions that mirror the hypothesized mathematical activity of the children as closely as possible" (p. 466).

The most relevant aspect of HI for this study is this operational definition for an LT. Based on this definition, the LT for length measurement (Clements et al., in press) articulates a developmental progression of increasingly sophisticated thinking for length measurement; therefore, it serves as a tool to measure children's conceptual and procedural knowledge for length measurement. In addition, the instructional tasks component of the definition of an LT provides an organizing structure for reflecting on the role that task play in revealing students' thinking as well as helping them progress through the levels. In the present study, this component lends itself to the reflection about the potential role of tasks involving comparing and measuring rectilinear and curvilinear paths for eliciting children's thinking about and possibly construct new and powerful understandings about key length measurement concepts.

A Learning Trajectory for Length Measurement

The LT for length measurement "describes an important sequence of knowledge about quantity, based on a ratio between a unit and the measured object, and other measured lengths as ratios" (Barrett, et al, 2012, p. 51). In the following sections I summarize the concepts and processes that define the levels of the LT for length measurement (Clements et al., in press). Across the first two levels of the length LT, children use continuous mental processes as they evaluate continuous extents. At the earliest level of the length LT, *Length Quantity Recognizer* (LQR), children identify length (the extent of an object from end-to-end) and distance (the amount of space between two points) as attributes; however, they do not yet understand length as a comparative. The second level, *Length Comparer* (LC), involves two sub-levels, *Length Direct Comparer* (LDC) *and Indirect Length Comparer* (ILC). At the LDC sublevel, children are able to physically align a pair of objects for the purpose of determining which is longer, and children at the ILC sublevel are able to use a third object to compare the lengths of two objects.

The transition into the third level of the length LT, the *End-to-End* (EE) level, marks a significant conceptual advance over the first two levels because it marks the development of the implicit concept that lengths can be composed of repetitions of shorter lengths. Students at this level understand that the number of repetitions of shorter lengths, or units, that fit along an object describe its length. Students at this level typically lay units end-to-end to measure the length of an object. At the *Length Unit Relater and Repeater* (LURR) level, children measure by repeating, or iterating, a unit. They also understand that more shorter units or fewer larger units are needed to measure the same object and can add two lengths to determine the length of a whole.

By the *Consistent Length Measurer* (CLM) level of the length LT, children are able to simultaneously imagine and conceive of an object's length as a total extent and a comparison of units. At this level of the length LT, children see length as a ratio comparison between the unit and the object measured. They measure straight paths consistently, use equal-length units, understand the zero point on the ruler, and can partition units to make use of units and subunits for the purpose of increasing precision. However, when determining the length of a bent path, children operating predominantly at this level may make rounding errors when measuring each segment and may not equate the sum of the parts of the bent path to the length of the whole. In addition, they may not be perturbed with geometric inconsistencies when coping with perimeter tasks. For example, when asked to draw a rectangle with a specified perimeter, a child at the CLM level may draw a rectangle with opposite sides that are not congruent. Children at this level apply multiplicative comparisons in simple situations, but typically rely on additive reasoning when making comparisons.

By the *Conceptual Ruler Measurer* (CRM) level of the length LT, children have an “internal” measurement tool. That is, they employ explicit strategies to estimate lengths reasonably, such as mentally iterating internal units of length or partitioning a length into equal-length parts. Children who are operating predominantly at the CRM level project or translate given lengths to determine missing lengths. When asked to draw a rectangle with a specified perimeter, children at the CRM level notice or are perturbed by geometric inconsistencies; they no longer accept rectangles with opposite sides that are not congruent. At this level, children increasingly use multiplicative reasoning in comparison situations.

At the *Integrated Conceptual Path Measurer* (ICPM) level, children are able to integrate and compare sets of units along each section of a bent path. When reflecting on the measure of a bent path or the perimeter of a polygon, they regard a group of units as a flexible object, a “string” of units wrapped around the entire perimeter or along the entire path. Therefore, in the context of a fixed perimeter or fixed path length task, children at the ICPM level are able to compensate for changes made to one side of a figure by adjusting other sides to maintain the fixed overall length. Although, they can find several related cases of polygons with the same perimeter, they may not yet be able to organize and synthesize a set of related polygons based on perimeter to formulate and justify a valid argument. At this level, children also begin to coordinate other measures with linear measures, such as curve, and show well-developed ideas about precision, such as constructing smaller units for the purpose of increasing precision.

The highest level of the current length LT is the *Abstract Length Measurer* (ALM) level. At this level, children have developed a continuous sense of length, and engage dynamic imagery to coordinate and operate internally on collections of units of units as well as collections of complex paths. Within the context of a fixed perimeter or path length task, they can synthesize sets of figures based on perimeter to formulate and justify a valid argument. Children at this level can coordinate multiplicative and additive reasoning in fluent ways and can engage in proportional reasoning about coordinated cases of paths for the purpose of reflecting on patterns among cases.

Summary: Relating Intuition to an LT for Length Measurement

The length LT describes a hierarchical sequence of knowledge about quantity, based on a ratio between a unit and a measured object. As children grow along the length LT, they develop sophisticated intellectual operations, or mental actions (concepts and processes). According to Fischbein's (1987) theory on intuition, children develop new intuitions as an effect of experience as well as the development of new intellectual operations; intuition is a developmental phenomenon. Therefore, informed by a synthesis of the two theoretical positions discussed in the sections above, this study was designed to explore developmental patterns of intuitive and analytical thinking for rectilinear and curvilinear path length for children who are operating at different levels of the LT for length measurement.

Review of the Related Literature

The LT for length measurement (Clements et al., in press) describes how children "establish rich conceptual knowledge of units of spatial measurement and use that knowledge as they measure in complex situations" (Barrett et al., 2011). It is a product of a line of research that has followed in the tradition of Piaget and his colleagues (1960). This length LT has been refined and revised over time using both cross-sectional (Barrett, Clements, Klanderman, Pennisi, Polaki, 2006; Clements et al., 1997) and longitudinal approaches (Barrett & Clements, 2003; Barrett et al., 2011; Barrett, et al., 2012; Clements et al., in press) and is rooted in prior research on children's thinking about length measurement concepts.

In the sections below, I first provide an overview of the body of literature from a about how children think and learn about length measurement concepts from a hierarchic interactionalist perspective (Sarama & Clements, 2009; Sarama, Clements, Barrett, Van Dine, & McDonel, 2011). Next, I describe the work of other teams of researchers that followed in the Piagetian tradition to produce alternative accounts to how children develop sophistication in conceptual and procedural knowledge for length measurement (e.g., Battista, 2006; Clarke, Cheeseman, McDonough, & Clarke, 2003). Finally, I conclude with a review of studies in mathematics education and psychology that address how children use intuitions for path length, and how those intuitions might interact with their conceptual and procedural knowledge for length measurement.

Children's Thinking about Length Concepts: A Developmental Perspective

Researchers in mathematics education have largely focused on children's thinking and learning about conceptual foundations of measurement: establishing a correspondence between a unit and an object to be measured, equal partitioning, the relationship between the size and number of units, the need for identical units, the iteration of same-size units, the accumulation of distance, and an understanding of the zero point on the ruler (Lehrer, 2003; Sarama & Clements, 2009; Stephan & Clements, 2003). This section focuses on studies in mathematics education that, from a hierarchic interactionalist perspective and taken together, suggest that children develop these key measurement concepts over time (Clements et al., in press; Sarama & Clements, 2009; Sarama, Clements, Barrett, Van Dine, & McDonel, 2011).

By Grade 2, most children develop an understanding of the inverse relationship between the size and number of units (Carpenter & Lewis, 1976; Lehrer, Jenkins, & Osana, 1998; Nunes & Bryant, 1996). For example, Nunes and Bryant (1996) found that some 5-yr old children and most 7-year old children could reason that two objects that are spanned with the same count of units, but different sized units, have a different measure.

Children in the primary grades exhibit difficulties with unit iteration (Ellis, Siegler, & Van Voorhis, 2003; Horvath & Lehrer, 2000; Lehrer, 2003). For example, Ellis, Siegler, and Van Voorhis (2003) found a significant age difference in the understanding of the concept of unit iteration from Kindergarten to Grade 2. Early on, children leave gaps or iterate with overlaps (Horvath & Lehrer, 2000; Lehrer, 2003). Researchers have also shown that children exhibit difficulties with a related concept, an understanding of the zero point (Lehrer, 2003; Stephan, Bowers, Cobb, & Gravemeijer, 2004). For example, when using a standard ruler, children often begin measuring from the tick mark labeled as “1” on a ruler (Lehrer, 2003). Similarly, when using nonstandard units, such as counting heal-to-toe steps, many students begin their count with their first movement (Stephan, Bowers, Cobb, & Gravemeijer, 2004).

Results from the National Assessment of Educational Progress (NAEP) indicate that the difficulties with unit iteration documented by researcher in mathematics education may persist beyond the primary grades. For example, when shown an image of a paper strip placed along a broken section of a ruler and asked to determine the length of the paper strip, 25% and 22% of Grade 4 students answered correctly for 2000 and 1996 NAEP, respectively (Kloosterman et al., 2004; Sowder et al., 2004). For a similar item, 40% and 63% of Grade 8 students answered correctly for the 2000 and 1996 NAEP, and 83% of Grade 12 students answered correctly for the 1996 NAEP. Although children exhibited higher percentages of correct responses across the elementary, middle, and secondary levels, these findings suggest that connecting numerical measurement with the process of unit iteration develops over time (Barrett & Clements, 2003; Battista, 2006; Clements, Battista, Sarama, Swaminathan, McMillen, 1997).

Most researchers in mathematics education have investigated children's developing conceptions for linear measurement in the context of straight or rectilinear paths. The task of determining the length of the curve has largely been regarded as a task that is beyond the scope of most K – 12 mathematics (Osborne, 1976). However, some researchers used the context of determining curve length to examine children's ability to operate on units and subunits and coordinate linear measure with another attribute, curve (see Clements et al., in press; Grugnett, Rizza, & Marchini, 2007). Specifically, Clements et al. (in press) showed that, when measuring a curve with a nonstandard unit, Grade 5 students exhibited strategies of fracturing the nonstandard unit to operate on subunits around the entire curve. Grugnetti, Rizza, & Marchini (2007) showed that, in an instructional setting, tasks involving approximating the length of a curve elicited elementary students' pre-conceptions of the limit. This suggests that, the task of determining the length of a curve could be a potentially fruitful context for investigating students' developing abilities to make sense of and use informal limit arguments by discussing processes in which a curve is represented by increasingly large numbers of segments of decreasing lengths to decrease the error in measuring and approach a true length of the curve.

Developmental Accounts for the Learning of Length Measurement

Using a developmental perspective, researchers in mathematics education have formulated models that describe how children's thinking and learning of length measurement concepts and procedures develops over time. Beginning with Piagetian theory (1960), in the following sections I describe and then compare and contrast these different developmental accounts for length measurement.

Piagetian theory. According to Piagetian theory, "[t]o measure is to take out of a whole one element, taken as a unit, and to transpose this unit on the remainder of the whole: measurement is therefore a synthesis of sub-division and change of position" (Piaget, Inhelder, & Szeminska, 1960, p. 3). Piaget described a developmental account of increasing sophistication beginning with *perceptual measurement* and culminating in *operational measurement*. Perceptual measurement, which is characterized by measuring using visual comparisons, "is inexact and merely appoximative, and it is subject to illusions or systematic errors (Piaget, Inhelder, & Szeminska, 1960, p. 29)." The process of evolution from perceptual measurement to operational measurement is complete when the child is capable of unit iteration, or “the construction of units to measure any distance in stepwise movement” (Piaget, Inhelder, & Szeminska, 1960, p. 30)

Piaget et al. (1960) specified a developmental account for the acquisition of increasing sophistication in the understanding of length measurement. This account distinguishes between an intuitive and *pre-operational* or perceptual conception of length measure and *operational composition*. Pre-operational children (Levels I and IIa and sub-stage IIb) do not yet understand the function of a unit of measure; however, operational children (Stages III and IV) conserve length and coordinate between subdivision and order of position. The following sections detail Piaget’s account of the development of measurement of length.

Levels I and IIa. At levels I and IIa, children do not yet conserve due to a lack of coordination between subdivision and change of position. That is, children at this level either subdivide without correctly applying the unit of measure or apply a change of position of the unit of measuring without adequately subdividing. At these stages, children have not yet constructed a unit and do not yet have transitivity; they rely mainly on visual inspection or motion along a path.

Sub-stage IIb. Sub-stage IIb is an intermediate stage. At this stage, conservation is “dimly perceived, and children at this level also begin to understanding transitivity in common measure, and later, even the role of a measuring unit” (Piaget, Inhelder, & Szeminska, 1960, p. 124). Children exhibit growth in terms of coordination, and progress toward “the beginnings of a synthesis of subdivision and relations of order and change of position” (p. 125). Understanding or sophistication is reached by trial-and-error. Understanding of transitivity may be pre-operational or intuitive here. Although children at this stage do not necessarily appreciate the need for exhaustively using same-size units, their understanding of a unit of measure increases through the process of trial-and-error.

Stage III. Stage III marks the transition to operational measure. At this stage, children coordinate subdivision and change of position; therefore, they are able to conserve. Within this stage, children at level IIIa exhibit evidence of operational transitivity without being able to subdivide a length into equal parts. At level IIIb, children have both operational transitivity and the capability of subdividing a length into equal parts, which is unit iteration.

Stage IV. At stage IV, children are capable of deductive composition. Children at this stage may initially engage in reasoning about specific cases using trial and error. Eventually, though, children at Stage IV may initially engage in actions that are experimental at first, their actions “eventuate in a reversible operational grouping, so coordinated as to yield universal generalizations which are deductive and necessary and which therefore transcend experience” (Piaget, Inhelder, & Szeminska, 1960, p. 208). That is, children at Stage IV are able to generalize from specific cases to form a logical deduction.

Following in the tradition of Piaget and his colleagues (1960), researchers have investigated how children develop sophistication in their thinking about length and measure over time (e.g., Barrett & Clements, 2003; Barrett et al., 2006; Battista, 2006; Clarke, Cheeseman, McDonough, & Clarke, 2003; Clements et al., 1997; Sarama & Clements, 2009). Clarke, Cheeseman, McDonough, and Clarke (2003) and Battista (2006) discussed the development of frameworks for the growth of children’s conceptions of length for the purpose of informing professional development. In each of these projects, the primary purpose of the framework was to inform formative assessment in the classroom. Sarama and Clements (2009) synthesized earlier work (e.g., Barrett & Clements, 2003; Barrett et al., 2006; Clements et al., 1997) into the length LT described above.

Early numeracy research project (ENRP) framework. The ENRP was a research and professional development project conducted in Australia in which teachers utilized a framework consisting of “growth points” in early mathematics learning (Clarke, Cheeseman, McDonough, & Clarke, 2003). The framework for length measurement, which was also meant to address mass, was informed by available literature (e.g., Brown et al., Dickson, Brown, & Gibson, 1984; Pengelly & Rankin, 1985; Wilson & Rowland, 1993), and used to develop assessment items to match each of the growth points. The classroom teachers who participated in the project conducted this assessment in an individual interview format with students in their own classrooms. Based on this assessment, particular growth points were assigned to the children. The framework for length (and mass) measurement consisted of five growth points:

1. At the first growth point (GP1), children show an awareness of the attribute of length and its descriptive language.
2. By the second growth point (GP2), children compare, order, and match objects by their lengths.
3. Next, at the third growth point (GP3), children appropriately use uniform units. That is, children are able to assign number and unit to the measure of length.
4. Children at the fourth growth point (GP4) choose and use formal units for accurately estimating and measuring length.

5. By the fifth and final growth point in this framework (GP5), children solve a range of problems that involve important concepts and skills that are related to length and its measure.

This growth points framework for length measurement was meant to provide “a sense of the typical order in which important understandings and skills develop” (p. 71).

Cognitively based assessment. In the United States, Battista (2006) designed a developmental account of elementary children’s thinking about length measurement based on his own empirical work within his Cognitively Based Assessment (CBA) project, which was a professional development project. Battista posed a two-part hierarchical account for the development of children’s length measurement concepts. Each part consists of levels, which describe cognitive plateaus that children reach as reasoning about length and measure evolves from “informal, pre-instructional reasoning to formal mathematical reasoning about length” (Battista, 2006, p. 141). This framework includes a 4-level account of the development of non-measurement reasoning and the second characterizes the development of measurement reasoning in 6 levels (Barrett & Battista, in progress). According to Battista’s levels of reasoning for length measurement, non-measurement reasoning “involves using visual judgments, direct comparisons, correspondences between parts, and transformations” (p. 141). Measurement reasoning then “involves determining the number of unit lengths that fit end to end along an object, with no gaps or overlaps” (Battista, 2006, p. 141). In this framework, non-measurement reasoning often emerges before measurement reasoning, but continues to develop even after measurement reasoning appears.

CBA and non-measurement reasoning. The account of non-measurement begins with **N0**. Children at this level rely on holistic visual comparisons. At the next level, **N1**, children correctly compare straight paths either using (a) direct comparison by placing objects next to each other or (b) indirect comparison by comparing objects using a third object. At the **N2** level, children manipulate or compare parts of complex paths in a systematic way; this level consists of two sublevels. In the first sublevel, students can rearrange pieces of paths to make new paths for the purpose of making comparisons. In the second sublevel, rather than transform one path into the other, students compare paths by matching same length pieces one-by-one in pairs. The second sublevel is a conceptual advancement from the first because, at the first sublevel, children rely on visual comparisons or manipulations, but at the second level they make inferences about the length of the entire path based on comparisons of the pieces of the paths (Battista, 2006). At the **N3** level, children make property-based transformations. That is, students make comparisons by transforming paths in ways that inform inferences based on geometric properties of shapes.

CBA and measurement reasoning. At the first measurement level, **M0**, children do not connect number to unit iteration. They often recite numbers while continuously moving their finger along a path or count dots without recognizing their count as an indicator of length. At the **M1** level, children attempt to iterate units, but initially do so incorrectly because they iterate with gaps, overlaps, or different size units. Eventually, they are able to iterate correctly along straight paths. By the **M2** level, children iterate correctly along all path types (straight, bent, and closed), and at level **M3**, they can operate on these iterations logically (by making inferences) and numerically (by adding, subtracting, multiplying, and dividing). At the **M4** level, children make property-based transformations, iterating unit lengths is not necessary because children can operate inferentially or numerically on length measurements. At the highest level, **M5**, children can understand and use formulas and variables. Children can understand and apply perimeter formulas and use variables in their reasoning about length (without referring to specific numbers).

Learning progressions in science education. Developmental accounts for the acquisition of knowledge of length and its measure are of interest to science educators as well as mathematics educators. Parallel to the LT construct that contributes to the conceptual framing of this study, science educators have outlined learning progressions (LPs) as "descriptions of successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic over a broad span of time (e.g., 6 to 8 years)" (National Research Council [NRC], 2007, p. 214). These LPs are critically dependent upon instruction (NRC, 2007). According to an NRC Committee on Science Learning (2007), LPs in science are anchored on one end by what is known about young children's reasoning and on the other end by societal expectations with respect to what older children should know about science. These progressions "are also constrained by research-based conceptual and social analyses of the structure of the disciplinary knowledge and practice that is to be learned" (p. 220).

A well-recognized LP in science education is the LP for the atomic-molecular theory of matter (LP for AMTM), which is a core idea in modern science, which was developed by Smith, Wiser, Anderson, and Krajcik (2006). The LP for AMTM describes a progression of more sophisticated answers to the questions regarding (a) what are things made of and how can one explain their properties, (b) what changes and what stays the same when things are transformed, and (c) how we know. Measurement plays an important role across all three questions. However, it is critically important for addressing the third question because, from this perspective, one learns about the world through measurement, modeling, and formulating and making sense of arguments (NRC, 2007, p. 364; see also Smith, Wiser, Anderson, & Krajcik, 2006).

According to Smith, Wiser, Anderson, and Krajcik (2006), measurement is a practice that is enabled by scientific knowledge. From this perspective, measurement involves ordering and quantifying. Ordering, or comparing along a dimension, involves going beyond categorization toward conceptualizing a continuous dimension, such as weight, temperature, hardness, or density. Quantifying encompasses measuring "important physical magnitudes such as volume, weight, density, and temperature using standard or nonstandard units" (Smith, Wiser, Anderson, & Krajcik, 2006, p. 8). The process of measuring itself "is a form of mathematical modeling and goes hand in hand with developing deeper conceptual understandings of the physical quantities in question" (p. 8). Smith et al. noted that many practices enabled by scientific knowledge are not limited only to the domain of science; they are the same practices that people use to "make sense of the world on everyday terms" (p. 9).

Smith, Wiser, Anderson, and Krajcik (2006) outlined three components of this big idea that are elaborated throughout the progression from Kindergarten through Grade 8: a) Good measurements provide more reliable and useful information about object properties than common-sense impressions, b) modeling is concerned with capturing key relations among ideas rather than surface appearance, and c) arguments use reasoning to connect ideas and data.

In Kindergarten through Grade 2, children learn important conceptual foundations of good measurement practices. That is, they learn that “good measurements use iterations of a fixed unit (including fractional parts of that unit) to cover the measured space completely (no gaps)” (NRC, 2007, p. 364). In Grades 3 through 5, children become aware that “measurements can be more or less precise and there is always some measurement error” (p. 365). Later, in Grades 6 through 8, children become aware that “sources of measurement error can be examined and quantified” and that “we can learn about properties of things using indirect measurement” (p. 365).

A Comparison of Developmental Accounts for Length Measurement

Table 1 below illustrates the relationships between the developmental accounts for children’s conceptions of length measurement according to Piagetian Theory (Piaget, Inhelder, & Szeminksa, 1960), ENRP (Clarke, Cheeseman, McDonough, & Clarke, 2003), CBA (Barrett & Battista, in press), and the length LT (Clements et al, in press) as well as the LP for AMTM (Smith, Wiser, Anderson, & Krajcik, 2006).

Table 1

A Comparison for Developmental Accounts for Length Measurement

Piagetian Theory	ENRP[1]	CBA: Non-Measurement Reasoning	CBA: Measurement Reasoning	LP for AMTM	Length LT
Level I and IIa subdivides without correctly applying unit or applies change of position without adequately subdividing; relies on inspection or motion along a path	**GP1** awareness of the attribute of length and its descriptive language	**N0** relies on holistic visual comparisons **N1** straight paths **N1a** direct comparison **N1b** indirect comparison	**M0** does not connect number to unit iteration		**LQR** understands length and distance as attributes; makes perceptual, intuitive comparisons **LDC** physically or mentally aligns objects to compare **ILC** compares lengths of two objects using a third object; applies transitive reasoning
	GP2 compares, orders, and matches objects by length	**N2** manipulates or compare parts of complex paths in a systematic way **N2.1** rearrange pieces of		**K – 2 Measurement** good measurements use iterations of a fixed unit	**EE** lays units end-to-end to measure linear extent (not necessarily using same-sized units); the number of these
	GP3				

Sub-stage IIb may not appreciate the need for same-size units; understanding of unit increases through trial-and-error	appropriately uses uniform units; assigns number and unit to the measure of length	paths to make new paths for the purpose of making comparisons		(including fractional parts of that unit) to cover the measured space completely (no gaps)	units that fit describes length **LURR** Iterates a single unit; begins to appreciate the need for same-size units; relates size and number of units; adds two lengths to obtain a whole
Stage III transitions to operational measure; coordinates subdivision and change of position; able to conserve		**N2.2** makes inferences about the length of the entire path based on comparisons of the pieces of the paths	**M1** attempt to iterate units, but initially do so incorrectly because they iterate with gaps, overlaps, or different size units; eventually iterates correctly along straight paths		**CLM** Recognizes the need for same-size units; recognizes the length of a bent path as the sum of its parts; partitions units to increase precision; may not coordinate size lengths along complex paths
Sub-stage IIIa have operational transitivity without ability to subdivide a length into equal parts **Sub-stage IIIb** have both operational transitivity and	**GP4** uses formal units for accurately estimating and measuring length		**M2** correct iteration on all paths **M3** operate on iterations by making inferences and by	**3 – 5 Measurement** measurements can be more or less precise; there is always measurement error	**CRM** operates with an “internal” measuring tool; fits smaller segments along curves to reduce overall error

the ability to subdivide a length into equal parts, which is unit iteration			adding, subtracting, multiplying, and dividing **M4** iterating unit lengths is not necessary because children can operate inferentially or numerically on length measurements		**ICPM** Shows well-developed ideas of precision and accuracy in selection of units; anticipates and monitors sets of related cases
Stage IV generalize from specific cases to form a logical deduction		**N3** make property-based transformations	**M5** can understand and use formulas and variables	**6 – 8 Measurement** sources of measurement error can be examined and quantified; we can learn about properties of things using indirect measurement	**ALM** Engages dynamic imagery to coordinate and operate internally on collections of units of units and collections of entire paths; construct and argue about derived units as a dimension; reasons about collection of measurements over time or across cases

Length Conservation

Aside from the Piagetian (Piaget, Inhelder, & Szeminksa, 1960) account of the learning of length measurement, the developmental accounts summarized in Table 1 above all articulate levels of sophistication for conceptual and procedural knowledge for length measurement. In Piaget and his colleagues' work (1960), the role of conservation of length, the recognition that length remains invariant under transformation, constrained concept growth for length measurement. According to Fischbein's (1987) theory on intuition, the ability or inability to conserve length is an intuitive apprehension. Therefore, Piaget's account of the learning of length measurement also attends to intuition as length conservation.

In the decades after Piaget and his colleagues (Piaget, Inhelder, & Szeminksa, 1960) published their findings, researchers in mathematics education (see Carpenter, 1975; Hiebert, 1981a, 1981b) showed that children's concept growth for length measurement is not constrained by the development of conservation. For example, in his work with some length conserving and non-conserving children in Grade 1, Hiebert (1981a, 1981b) investigated the effect of instruction on some of the key conceptual foundations of length measurement, such as unit iteration and the relationship between the size and number of units, on their ability to conserve length. He found that children's ability to iterate units of length had no impact on their ability to conserve; however, recognition of the relationship between the size and number of units needed to measure a length was related to conservation.

Results from research have indicated that there exists a general lack of relationship between the conservation of length and understanding of length measurement concepts and procedures (Clements, 1999; Lehrer, 2003). Therefore, for the case of length conservation, research has shown that there exists a general lack of relationship between intuition for path length and conceptual and procedural understanding of length measurement.

Intuitions for Path Length

Researchers have explored students' intuitions about length in the contexts of rectilinear paths (Barrett & Clements, 2003; Chiu, 1996). Chiu (1996) explored sixth grade students' origins, uses, and interactions of students' intuitions in the context of comparing rectilinear paths. In keeping with Fischbein's (1987) characterization of intuition, he defined intuitions as "self-evident notions that are robust, holistic, and conceptual" (Chiu, 1996, p. 479). First, intuitions are robust because they are applicable in many situations and alternatives are not plausible. Intuitions are holistic because they retain meaning only as a whole. Finally, intuitions are conceptual because creating and/or applying an intuition requires conceptualizing beyond just immediate perception.

The sixth grade students in Chiu's (1996) study repeatedly used a limited number of intuitions, which originated from their everyday experiences: compression, detour, complexity, and straightness. Students who used the compression intuition discussed the unfolding or straightening of the path and referred to a path as being longer than it seems because it is compressed. The detour intuition appeared as students discussed a path in terms of its wandering away from the destination or doing something else instead of moving toward it. The complexity intuition emerged as students attended to the number of components such as segments or turns when comparing rectilinear paths. Students who relied on the straightness intuition chose a particular path as the shortest because it was straighter than another path without providing justification. Chiu distinguished these four intuitions from analytic procedures and complex algorithms. An analytic procedure, such as using a ruler to measure each path and comparing the lengths, or applying an “align-and-compare algorithm” (Chiu, 1996, p. 485), which involves projecting corresponding horizontal and vertical segments, are not intuitions.

Chiu (1996) posed one rectilinear path length comparison task within each of two problem-solving sessions. In the initial session, he first posed the tasks of ranking the lengths of three rectilinear paths. For children who did not solve the problem, he provided access to a ruler, graph paper, index cards, paper clips, rubber bands, string, scissors, and tacks to afford the opportunity for the child to get perceptual feedback. If a child did not use the align-and-compare algorithm, he encouraged the child to construct the algorithm through the use of guiding questions. He found that every child used at least one intuition. Many of the children used a variety of intuitions when comparing and ranking rectilinear paths by length; these intuitions supported one another in some instances and provided conflicting information in others. Even after being taught an applicable algorithm for comparing sets of rectilinear paths by length, they first used their intuitions before applying the algorithm. Chiu concluded that middle school students’ intuitions for rectilinear paths were sparsely connected and coexisted with standard mathematical knowledge, such as the align-and-compare algorithm.

Chiu (1996) regarded these intuitions not as misconceptions, but as productive knowledge pieces. He suggested that children “may learn more by assessing them with more sophisticated criteria, such as range of applicability, ease of use, and coherence with other ideas” (p. 500). Many of the children applied a variety of intuitions to solve the problems that Chiu posed; therefore, he argued for instruction that helps children "develop tools to coordinate and elaborate them, thereby avoiding indecision and capitalizing on learning opportunities" (p. 500). Chiu suggested that children's intuitions for path length serve as an important foundation for mathematical concepts. For example, the detour and straightness intuitions are particularly powerful for understanding the triangle inequality theorem. Imagining decomposing a triangle into two paths, a straight path consisting of a single segment and a bent path consisting of two segments, and comparing those paths by length can help children link their path length intuitions to formal mathematics.

In their work, Barrett and Clements (2003) found evidence of intuitive thinking for path length when investigating children's developing abstractions for linear measurement. Over the course of their six-month teaching experiment with four children in Grade 4, students were presented a task involving a 24-unit notched straw manipulative to make rectangles and triangles that had a perimeter of 24, and drew records of the rectangles and triangles they had made with the straw manipulative. When asked to respond to a fictitious student who had double counted corner tick marks in a drawing of a straw triangle, one of the four students, Alex, said that tick marks at corners should be counted twice because "corners count for more." Alex explained that paths with more corners are longer because one must turn more when traversing them. Barrett and Clements (2003) noted that Alex's explanation is evidence of Chiu's (1996) description of intuitive thinking for path length, which is deeply ingrained and based on children's informal experiences. Alex's response suggests that his developing abstractions for linear measurement, with respect to establishing exact correspondence between counting and linear dimensions of paths, interact with his intuitive thinking for path length.

Psychological Foundations of Intuitions for Path Length

Studies in mathematics education (i.e., Barrett & Clements, 2003; Chiu, 1996; Clements, Battista, Sarama, & Swaminathan, 1996; Mitchelmore, 1997) and psychology (i.e., Montello, 1997; Pressey, 1974; Thordyke, 1981;) have explored the psychological foundations of intuitions for path length. The complexity intuition (Chiu, 1996), observed by an attention to the number of segments or turns when comparing rectilinear paths, has been documented by psychologists across several studies. Allen (1981) and Montello (1997) documented the "route segment" hypothesis, which they described as people's tendency to provide longer estimates for paths that are partitioned into several separate segments. Sadalla and Staplin (1980) observed a "clutter effect" on people's judgments or estimates for path length. In their study, people who crossed several intersections estimated length to be longer than people who crossed fewer intersections. Byrne (1979) found that people tend to overestimate lengths of routes with a greater number of bends. Thorndyke (1981) observed a similar phenomenon. In his study, he concluded that people overestimate length of routes with a greater number of intervening points. All of these studies suggest that the complexity intuition is robust across a wide age range and across a wide variety of contexts (e.g., Thordyke, 1981; Kosslyn, Pick & Fariello, 1974; Luria, Kinney; & Weissman, 1967; Pressey, 1974).

CHAPTER III

METHODOLOGY

Introduction

In this chapter, I discuss the design of the qualitative research methodology that guided the study. I then describe participant selection and data collection procedures. Next, I discuss the design and coding, as well as procedures used for measuring the validity of a participant selection instrument, a written length measurement assessment. This is followed by a section in which I describe how interview participants were selected and interview data were collected. Finally, I explain the design of the interview tasks, highlighting the purpose for including each task, the methods used for analyzing students' responses for the interview data, and the procedures used for the frequency analysis that informed elaborations to the current hypothetical learning trajectory (LT) for length measurement (Clements et al., in press).

Overview of the Study Design and Procedures

The study seeks to relate students' intuitive and analytical thinking for path length to an LT for length measurement (Clements et al., in press). Specifically, this study explores the intuitions and analytical strategies that elementary, middle, and secondary students use when comparing rectilinear as well as curvilinear paths in two-dimensional space by length, which has not been addressed in prior studies. This study is exploratory in nature; therefore, I planned it according to a basic qualitative research design

(Merriam, 2009) so that I would be able to follow new or unexpected themes present in the data. The design of this study makes use of a written length LT-based assessment administered to a sample of students and structured, task-based interviews with a subset of the sample (Goldin, 2000). The design of this study was informed by methods used in previous research that was focused on extending LTs for length (Beck, Eames, Cullen, Barrett, Clements, & Sarama, 2014), volume (Kara, 2013), and area measurement (Cullen, Miller, Witkowksi-Rumsey, Barrett, & Sarama, 2011). These studies made use of a similar methodological organizing structure for extending an LT. Key elements of this method of extending an LT include a) designing tasks that reveal student thinking for an aspect not addressed in the LT, b) presenting those tasks to a sample of students that include some students at the same LT levels and some at adjacent LT levels, c) describing and differentiating students' responses to each task, and d) comparing the strategies of students within the same LT level and across adjacent LT levels to inform recommendations for extensions to the LT.

In the present study, the aspect not addressed in the LT were student's intuitions and analytical strategies for comparing sets of rectilinear or curvilinear paths. I designed task-based interviews (Goldin, 2000) to reveal students' intuitive and analytical thinking for rectilinear and curvilinear paths. In addition, I used a written length LT-based assessment, which was designed to probe students' thinking at different levels of the length LT so that the level best describing each participant's conceptual and procedural knowledge for length measurement could be identified for the purpose of recruiting a sample of students that include some students at the same LT levels and some at adjacent LT levels. I coded students' responses to the written LT-based assessment using the length LT. I described and differentiated students' responses to the structured, task-based interviews (Goldin, 2000) using an existing coding scheme (Chiu, 1996) and a constant comparative method (Corbin & Strauss, 2008). I then compared strategies among students within the same level and across adjacent levels to inform recommendations for extending the length LT with respect to students' intuitions and analytical strategies for rectilinear and curvilinear path length.

Because one of the goals of this study is to extend the literature on children's conceptions of length measurement beyond elementary aged children to include middle and secondary school students, I recruited participants from Grades 4, 6, 8, and 10 as both a convenient and purposeful sample. The written LT-based assessment (Appendix A) was administered as part of regular classroom activities to all of the students in each grade included in the study for the purpose of probing students' thinking at different levels of the length LT and identifying the level that best described each student's level of sophistication for length measurement. The length LT level placements attributed to each of the students in the entire sample informed the selection of a subset of these students to participate in two individual, task-based interviews designed to probe students' intuitive and analytical thinking for rectilinear and curvilinear path length.

Participant Selection and Data Collection Procedures

Participants and Context for Research

The sample consisted of 82 consenting students: 22 each in Grades 4 and 6, 20 in Grade 8, and 18 in Grade 10. I recruited participants from two different private schools in the Midwest, one for pre-K – 8 students and another for pre-K – 12. At the pre-K – 8 school, I selected participants from two classes each in Grades 4, 6, and 8. I selected the

18 consenting Grade 10 participants from the pre-K – 12 school where there were a total of 22 Grade 10 students enrolled in Algebra I, Algebra II, and advanced math.

The pre-K – 8 school, from which I recruited the students in Grades 4, 6 and 8 offers an academic program that includes five core subjects: language arts, math, reading, science, and social studies. Classes in religion, physical education, art, computers, choral music, and instrumental music are also included.

The pre-K – 12 school, from which I selected the Grade 10 students is an independent liberal arts, college-preparatory school. Students at all levels take six core subjects: Bible, history, English, science, math, and foreign language. In addition, students in Grades 9 through 12 may elect to take art, music, physical education, or technology to supplement the six core subjects. Dual-credit and on-line courses are also made available to them.

Data Collection Procedures

I administered a participant selection instrument, a written length LT-based assessment, to all Grade 4, 6, and 8 students at the pre-K – 8 school and Grade 10 students at the pre-K – 12 school. I coded assessments for the 82 consenting students using the levels of the length LT; these coding procedures and methods for analysis are described in the sections below. Based on the results of this assessment, I recruited a subset of 16 students, who represented the four grade levels and four length LT levels relevant to the present study, to participate in two structured task-based interviews.

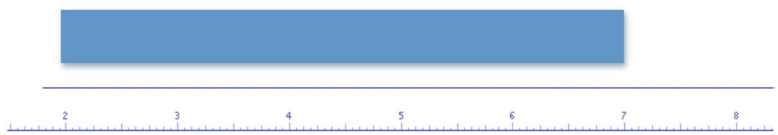
Participant Selection Instrument: Design and Coding

Prior to the study, I anticipated that most of the students across Grades 4 through 10 would be operating within the Consistent Length Measurer (CLM), Conceptual Ruler

Measurer (CRM), Integrated, Conceptual Path Measurer (ICPM), and Abstract Length Measurer (ALM) levels of the length LT. Therefore, I selected items, which were initially developed and refined through a process of piloting within NSF-funded projects aimed at studying elementary (DRL 0732217) and middle school (DRL 1222944) students' conceptions of spatial measurement, to be accessible to students within the these four levels.

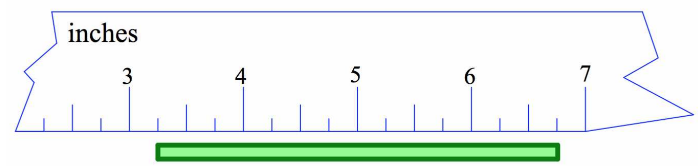
Each assessment task was designed to elicit observable strategies that are indicative of particular mental actions and objects that differentiate the levels of the established length LT (Clements et al., in press). Some of the tasks included in the written LT-based assessment were designed to reveal thinking at a variety of LT levels. For the purpose of designing the LT-based assessment for this study, I mapped tasks to the highest length LT level of thinking they have been shown to elicit in prior research (DRL 0732217; DRL 1222944). To provide confidence in the level placement assigned by this instrument, I included two items each for the CLM (see Figures 1 and 2) and CRM levels (see Figures 3 and 4). Because prior research has documented difficulties with designing items that can differentiate students at the highest levels of the length LT (Clements et al., in press), I included a set of three items to probe students' thinking at the ICPM and ALM levels (see Figures 5, 6, and 7). The following sections describe my design, the purpose of including each task, as well as the methods or procedures that I used to analyze students' responses to the written length LT-based assessment.

CLM level items. Assessment Tasks 1 and 2 shown in Figures 8 and 9 below have been shown to elicit thinking at the EE, LURR, or CLM levels of the length LT (Barrett et al., 2012). Therefore, in the present study, I regarded them as CLM-level items. (Note: The actual length LT-based assessment in the form that was administered to the Grade 4, 6, 8, and 10 classes is included in Appendix A.)



Using the drawing of a part of a ruler as a guide, measure the strip of paper shown above it. How many inches long is the strip?

Figure 8. Written LT-based assessment CLM level item, Task 1.

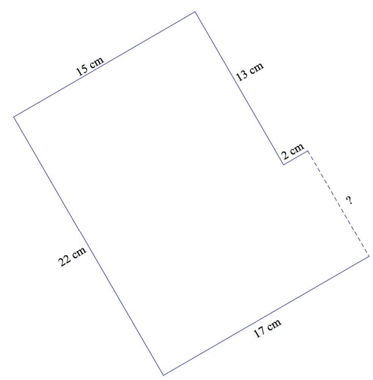


This is a picture of a rod just below a broken section of a ruler. Use this picture to measure the length of the rod. How long is the rod?

Figure 9. Written LT-based assessment CLM level item, Task 2.

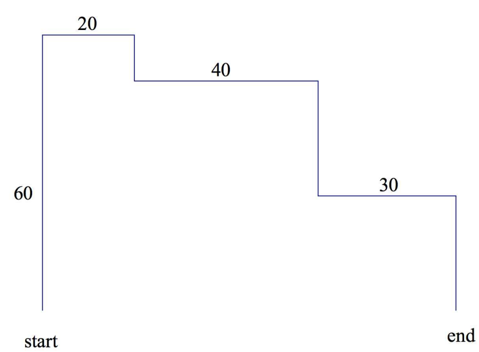
The CLM level items shown in Figures 8 and 9 were designed to investigate students' ability to integrate intervals and endpoints of those intervals (Barrett et al., 2012; Cullen, 2009). For example, when resolving the misaligned paper strip item in Figure 8, students who report the length of the misaligned paper strip as 7, the number corresponding to the endpoint, have developed the implicit concepts that objects can be composed of smaller objects and that a count of those objects can represent a measure of an attribute of an object. However, they have not yet developed the concept of unit iteration; this is consistent with EE-level thinking. Students, who incorrectly count tick marks and report the length of the paper strip as 6, have begun to develop the concept of unit iteration. This tick mark counting strategy is indicative of LURR-level thinking. Children who correctly resolve this task by counting intervals, correctly counting tick marks at the end of each interval, or operating arithmetically on measures (i.e., computing 7 – 2) and answer 5, show that they see a measure as a ratio comparison between an object and a unit, and they have a well-developed concept of unit iteration. This is consistent with CLM level thinking. Therefore, I considered this item to be a CLM level item when designing the written length LT-based assessment. Task 2 (Figure 9), which involves fractional units, also probes students' capabilities for maintaining this integration of intervals and endpoints for units, inches, and subordinate units, quarter inches.

CRM level items. Assessment Tasks 3 and 4 (Figures 10 and 11) have been shown to indicate whether students are at the CRM level of the length LT or are not yet at CRM (Clements et al., in press).



Find the measure of the missing side length.

Figure 10. Written LT-based assessment CRM level item, Task 3.



Find the length of the total path, from start to end, shown above.

Figure 11. Written LT-based assessment CRM level item, Task 4.

These two tasks were designed to explore students' capabilities for projecting or translating given lengths to determine missing lengths (Clements et al., in press) in the context of a rectilinear figure, Task 3 (Figure 10), and a rectilinear path, Task 4 (Figure 11). A correct numerical response of 9 for Task 3 or 210 for Task 4 indicates that a student is capable of projecting or translating given lengths to determine one or more missing lengths, which is consistent with CRM level thinking. An incorrect response indicates that a student is not yet at the CRM level.

ICPM and ALM level items. Assessment Tasks 5, 6, and 7 (Figures 12, 13, and 14) have been shown to be accessible to children at the CLM, CRM, ICPM, and ALM levels of the length LT (DRL 0732217; DRL 1222944). Specifically, Tasks 5 and 6 were designed to explore students' ability to find several related cases of polygons with the same perimeter and to relate those cases to one another by logical comparison, which is ICPM level thinking (Clements et al., in press). Task 6 also reveals students' abilities for coping efficiently and precisely with subordinate units in the context of finding related cases of polygons with the same perimeter. Part b for both items 5 and 6 also have the potential to reveal whether students are aware that subdividing a unit into subunits is a process that is potentially unlimited, which is ALM-level thinking.

Imagine making an L-shaped path from a string that is 10 cm long.

a. How many different L-shaped paths would you be able to form in all?

b. Use the space below to explain how you got your answer and why you think your answer is correct.

Figure 12. Written LT-based assessment ICPM and ALM level item, Task 5.

a. Use the space below to sketch **two different** rectangles, each having a perimeter of 2 inches. For each of your rectangles, label the lengths of all four sides.

b. How many **more** rectangles have a perimeter of 2 inches? _____________

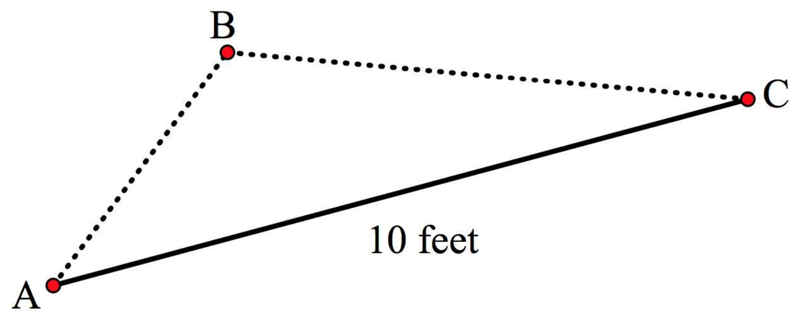
Figure 13. Written LT-based assessment ICPM and ALM level item, Task 6.

For Tasks 5 and 6, drawings that reflect geometric inconsistencies, such as a rectangle with opposite sides labeled as different lengths, indicate that a student does not yet coordinate linear extent with geometric properties. This is consistent with the CLM level. Students who provide drawings that are not geometrically inconsistent but do not show evidence of coordinating a *set* of comprehensive cases exhibit the CRM level. Responses that reflect several related cases of paths with the same length or polygons with the same perimeter, as well as evidence of relating those cases to one another by logical comparison, indicate an ability to conceive of a group of units as a flexibly wrapped string along the length of a path or perimeter of a polygon; this is consistent with the ICPM level (Clements et al., in press). Responses that reflected a synthesis of sets of paths with a fixed length or polygons with a fixed perimeter, including those with non-integer segments or side lengths, to formulate and justify an argument, while attending to the potentially unlimited process of subdividing units, exhibit the ALM level (Clements et al., in press).

The final task included on the written length LT-based assessment, Task 7, was designed to assess students' ability to coordinate geometric properties, such as angle, with linear extent. These mental actions are consistent with the highest level of the current LT for length measurement (Clements et al., in press). Task 7 is shown in Figure 14 below.

You need to bury a wire in your backyard that connects points A and C. One option is to run a 10-foot wire directly from points A and C, which is indicated by the solid line in the picture below. Another option is to run a wire from point A to C through point B, which is indicated by the dotted line.

We know that points A and B are 10 feet apart. However, no one measured the length of the path from A to C through point B (the dotted line).



a. Use the space below to explain how *long* you think the wire will need to be to connect points A and C through B.

b. Use the space below to explain how much wire you will *buy* so that you can be sure you have enough to connect points A and C through B.

Figure 14. Written LT-based assessment ICPM and ALM level item, Task 7.

Written Length LT-based Assessment: Task-by-Task Analysis

I coded each student's response for each of the seven items of the written length LT-based assessment using the levels of the length LT, based on the observable strategies used to generate a solution. Because the levels of the length LT are described in terms of the observable strategies and corresponding mental actions or objects, I then used these strategies to assign a length LT level claim for each student for each of the seven tasks on the assessment instrument. I coded students' observable strategies, which were not consistent with any of the levels of the length LT, as "No Claim."

I tracked the distribution of the level claims for each task within and across each grade. I then compared the distributions of level claims for conceptually congruent tasks, such as the pair of CLM items, Tasks 1 and 2 (Figures 9 and 10), for the purpose of describing the validity of the items with respect to assessing the mental actions and objects associated with the intended length LT level for the grades included in the study.

I coded students' responses to Tasks 1 and 2 as EE, LURR, or CLM. The distribution of levels exhibited by the 82 Grade 4, 6, 8, and 10 students' responses for Tasks 1 and 2 (see Figures 9 and 10) are shown in Figure 15 below.

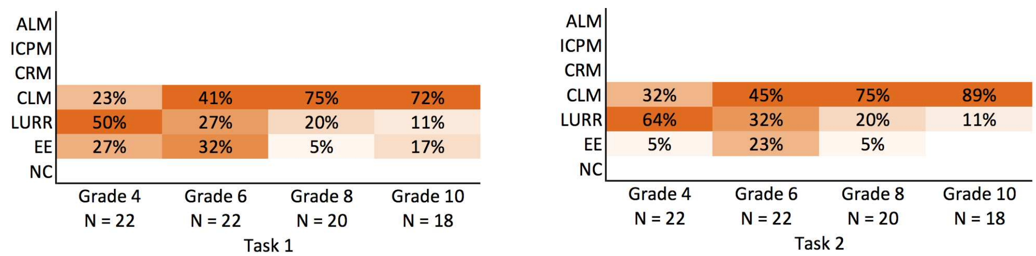


Figure 15. Distribution of length LT levels for Tasks 1 and 2.

Figure 15 shows that the distribution of level placements within each grade is generally consistent across Tasks 1 and 2. The percentage of students within each grade who exhibited CLM level thinking on Tasks 1 and 2 increased from Grades 4 to 6 and again from Grades 6 to 8. However, the increase in frequency of CLM level thinking remained consistent across Grades 8 and 10. Fewer instances of EE level thinking were observed on Task 2, the fractional broken ruler task, than on Task 1, the integer broken ruler task.

I coded students' responses for Tasks 3 and 4 (see Figures 10 and 11) as either CRM or not yet CRM. Figure 16 below illustrates the distribution of students' responses to Tasks 3 and 4 within each of Grades 4, 6, 8, and 10.

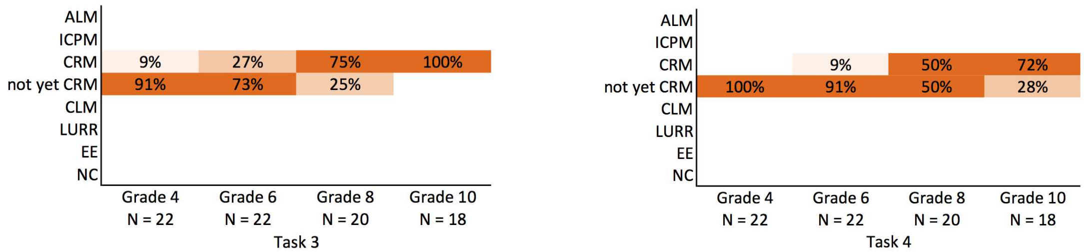


Figure 16. Distribution of length LT levels for Tasks 3 and 4.

The distribution of “CRM” and “not yet CRM” level thinking within each grade is largely consistent across Tasks 3 and 4. However, within each grade, fewer students used CRM level thinking on Task 4 than on Task 3. Figure 16 illustrates a pattern of increased instances of CRM level thinking across Grades 4, 6, 8, and 10. That is, at higher grade levels, higher percentages of students used CRM level thinking to resolve Tasks 3 and 4.

I coded students’ repsonses on Tasks 5 and 6 as No Claim; CLM; CRM; not yet ICPM; (ICPM), which indicates some evidence of ICPM level thinking; ICPM; and ALM. Figure 17 below shows the distribution of level claims for Tasks 5 and 6 within Grades 4, 6, 8, and 10.

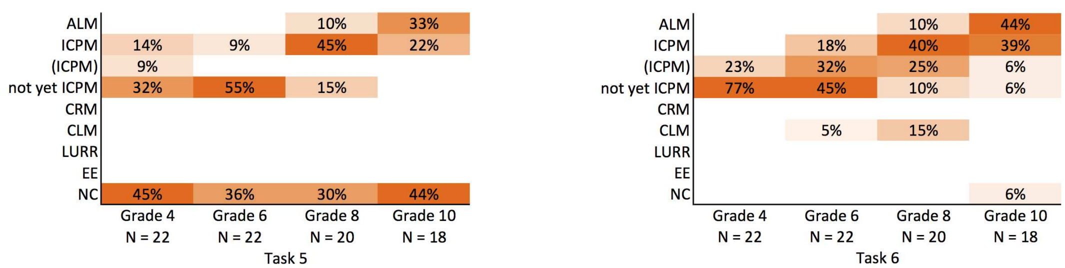


Figure 17. Distribution of length LT levels for Tasks 5 and 6.

Between 30 and 45% of students’ responses within each grade were coded as “No Claim” for Task 5; whereas, only 6% of students in one single grade level, Grade 10, were coded as “No Claim” for Task 6. This suggests that Task 5 may not be a valid task for eliciting students’ thinking at the CLM, CRM, ICPM, and ALM levels of the length LT. For Tasks 5 and 6, instances of the ALM level did not emerge until Grades 8 and 10, with a higher percentage of occurrences in Grade 10. Tasks 5 and 6 showed that children as young as Grade 4 showed evidence of ICPM level thinking. For Task 6, instances of “not yet ICPM” decreased across Grades 4, 6, 8, and 10. Claims of some evidence of ICPM level thinking, denoted as “(ICPM)” in Figure 17, increased from Grade 4 to 6. Full placement at ICPM, denoted as “ICPM” in Figure 16, increased from Grade 6 to 8, and remained stable across Grades 8 and 10. Instances of the ALM level increased from Grades 8 to 10.

I selected Task 7 on the written length LT-based assessment (see Figure 14) to reveal students’ thinking at the ALM level of the length LT; however, students’ responses to this task yieded codes of “No Claim” in most instances. Therefore, I did not consider this task for further for analysis within and across Grades 4, 6, 8, and 10.

Written Length LT-based Assessment: The Distribution of Level Placements

Based on the collection of seven tasks, which made up the written length LT-based assessment, I made a predominant length LT level claim for each of the 82 participants in the sample (22 students in Grades 4 and 6, 20 in Grade 8, and 18 in Grade 10). I tracked the distribution of aggregate level claims within and across the grade levels for the purpose of comparing the performance of each interview participant to their peers. Figure 18 below illustrates the distribution of these level placements within each grade.

	Grade 4 N = 22	Grade 6 N = 22	Grade 8 N = 20	Grade 10 N = 18
ALM	0%	0%	0%	56%
ICPM	0%	5%	50%	22%
CRM	0%	27%	30%	22%
CLM	41%	27%	10%	0%
LURR	32%	18%	5%	0%
EE	27%	23%	5%	0%

Figure 18. Distribution of predominant length LT levels within each grade.

In Grade 4, all of the students placed in the EE, LURR, and CLM levels of the LT for length measurement. Most of the students in Grade 6 exhibited EE, LURR, and CLM level thinking, but some students showed evidence of growth into the CRM (27%) and ICPM (5%) levels. In Grade 8, most of the students showed evidence of CRM (30%) and ICPM (50%) level thinking; however, the lowest 20% of the class still operated at the EE (5%), LURR (5%), and CLM (10%) levels of the length LT. By Grade 10, none of the students predominantly exhibited EE, LURR, or CLM level thinking on the written assessment tasks. Over half of the Grade 10 students placed at the ALM level with the rest of the students at the ICPM (22%) and CRM (22%) levels.

Written LT-based Assessment: Four Length LT Level Groups

From the sample of 82 students, a subset of 16 participants, who were operating predominantly at the CLM, CRM, ICPM, or ALM levels, was selected for two individual interviews. Students whose responses on the written length LT-based assessment that were clearly identifiable using the length LT, and whose aggregate level claim reflected a placement at CLM, CRM, ICPM, or ALM were considered for the interviews. Table 2 shows each of the 16 interview participant's predominant LT level placement in relation to the distribution of predominant length LT levels within his or her grade (Figure 18).

Table 2

Interview Participants' Length LT Placements Relative to Grade Level Distribution

Name	Predominant Length LT Level Placement	Grade	Length LT Placement Relative to the Distribution of Length LT Levels
Mia	CLM	Grade 4	Upper 41%
Kevin	CLM	Grade 4	Upper 41%
Noah	CLM	Grade 4	Upper 41%
Jenny	CLM	Grade 4	Upper 41%
Trent	CRM	Grade 6	Upper 32%
Ned	CRM	Grade 6	Upper 32%
Rose	CRM	Grade 6	Upper 32%
Lynn	CRM	Grade 8	Lower 50%
Grant	ICPM	Grade 6	Upper 5%
Rick	ICPM	Grade 8	Upper 50%
David	ICPM	Grade 8	Upper 50%
Ruth	ICPM	Grade 8	Upper 50%
Zane	ALM	Grade 10	Upper 56%
Scott	ALM	Grade 10	Upper 56%
Marie	ALM	Grade 10	Upper 56%
Kyle	ALM	Grade 10	Upper 56%

I sought to evenly represent the four grade levels and the four length LT levels, with two girls and two boys within each of the four length LT levels. Because the only ALM level placements came from Grade 10 students, I first chose four Grade 10 ALM level participants. From there, I chose interview participants to form groupings of two girls and two boys operating predominantly at each of the CLM, CRM, and ICPM levels from Grades 4, 6, and 8. Because only 41% of Grade 4 students were at the CLM level, 33% of Grade 6 students were at the CRM level and above, 50% of Grade 8 students were at the ICPM level, and 56% of students were at least at the ALM level, most of the students recruited to participate in interviews performed in the top half of their grade on the written LT-based assessment (see Figure 18).

Task-Based Interviews: Task Design

Each of the 16 interview participants was interviewed individually on two separate occasions through structured, task-based interviews (Goldin, 2000). The interviews consisted of a participant (student) and an interviewer interacting in relation to the tasks introduced to the student using a scripted protocol (Goldin, 2000), which I refined through pilot work.

A total of 10 tasks were spread across two interviews, with five tasks included in each interview. The protocols for each interview are included in Appendix B. The duration of each interview varied from 20 to 30 minutes. The time between the two interviews for each individual student was less than three weeks. I presented each participant with the same tasks across those sessions in the same order. The interviews took place in the school building during class time. I asked the Grade 4, 6, 8, and 10 teachers for the best time to interview the students to minimize interruption of normal classroom activities. Table 3 below summarizes the schedule for interview data collection.

Table 3

Interview Data Collection Schedule

Pseudonym	Interview 1		Interview 2	
Mia	10/14/13	1:00 PM	10/21/13	1:00 PM
Kevin	10/14/13	1:30 PM	10/21/13	1:30 PM
Noah	10/14/13	2:15 PM	10/21/13	2:15 PM
Jenny	10/14/13	2:45 PM	10/23/13	2:45 PM
Lynn	10/15/13	9:15 AM	10/22/13	9:15 AM
Rose	10/15/13	10:45 AM	10/22/13	10:45 AM
Trent	10/15/13	1:45 PM	10/22/13	1:45 PM
Ned	10/16/13	1:45 PM	10/23/13	1:45 PM
David	10/15/13	12:15 PM	10/22/13	12:15 PM
Rick	10/16/13	9:15 AM	10/23/13	9:15 AM
Grant	10/16/13	10:00 AM	10/23/13	10:00 AM
Ruth	10/16/13	12:15 PM	10/23/13	12:15 PM
Scott	11/4/13	8:55 AM	11/5/13	8:55 AM
Zane	11/4/13	9:20 AM	11/5/13	2:30 PM
Marie	11/4/13	12:50 PM	11/5/13	12:50 PM
Kyle	11/4/13	1:20 PM	11/5/13	1:20 PM

Structured, task-based interviews, involve an interviewer and participant(s) interacting within one or more scripted, preplanned tasks. The goal in a structured, task-based interview is to "observe, record, and interpret complex behaviors and patterns in behavior, including subjects' spoken words, interjections, movements, writings, drawings, actions on and with external materials, gestures, facial expressions, and so forth" (Goldin, 2000, p. 527). Because student thinking, reasoning, cognitive processes, internal representations, or knowledge structures cannot be directly observed, the aim of a task-based interview is to produce observable outcomes that can inform inferences about students' thinking. The primary purpose for including structured, task-based interviews in this study (a total of 32) was to address the research question with respect to exploring the intuitions and analytical strategies that elementary, middle, and secondary school students use when comparing rectilinear and curvilinear paths by length.

The exploration of student thinking during structured, task-based interviews proceeds according to four stages (Goldin, 2000). First, the question is posed and time is allowed for the child to respond. The interviewer responds with a nondirective follow-up, such as “Please, tell me more about that.” The second stage, in the event that the response from the subject is not spontaneous, the interviewer responds with minimal heuristic suggestions, such as “Could you show me using some of the materials on the table?” The third stage proceeds in the event that the description requested from stage 2 does not occur; this stage involves the guided use of a heuristic suggestion, such as “Do you see a pattern in the cards?” The fourth and final stage involves questioning that is exploratory and metacognitive in nature, such as “Could you explain how you thought about the task?” At each stage, the interviewer’s goal is to elicit “a complete, coherent verbal reason for each of the child’s responses, and a coherent external representation constructed by the child” (Goldin, 2000, p. 523).

I selected interview tasks to elicit observable evidence of students’ intuitive and analytical thinking for rectilinear and curvilinear paths as statements, gestures, and manipulations of tools. To draw out intuitive or analytical thinking, I asked students to compare sets of rectilinear paths and curvilinear paths without tools (Tasks 1, 2, 6A, 7, and 8A). Because of the scant body of research with respect to students’ thinking about curvilinear paths in two-dimensional space, I posed tasks that involved comparing a straight object and a curve (Tasks 3, 4, and 5), indirectly comparing two curves using a straight object (Tasks 6B and 8B), or measuring curves (Tasks 9 and 10). I posed the same tasks in the same order to all students, and I selected tasks to be accessible to students at the CLM, CRM, ICPM, and ALM levels of the LT. Across the tasks, I varied the representation of the unit: no tool, a 4-inch stick, or a standard ruler. In addition, I varied the paths according to intuitive interference, such as the number of turns, deviation from endpoint (Chiu, 1996), and tightness of curve. The design process described above yielded four categories of conceptually congruent tasks, which I describe in Table 4 below.

Table 4

Summary of Classes of Conceptually Congruent Interview Tasks

Task Category Description	Order Appearing in Interviews
Comparing sets of rectilinear paths by lengths	Tasks 1 and 2
Comparing sets of curvilinear paths by lengths	Tasks 6A, 7, and 8A
Comparing curves and a straight object	Tasks 3, 4, 5, 6B, and 8B
Measuring a curve with a standard ruler	Tasks 9 and 10

The protocol for each interview in Appendix B contains a complete description of the implementation of each task. See Appendix C for images of the actual size that were given to students during the interviews.

Overview of Interview 1 Tasks

The first interview consisted of two rectilinear bent path comparison tasks, Interview Tasks 1 and 2 (Chiu, 1996), and three tasks that involved comparing a curve and a straight object, Interview Tasks 3, 4, and 5 (Clements et al., in press).

The purpose of including the two rectilinear bent path comparison tasks (Chiu, 1996) was to probe students' intuitive and analytical thinking for comparing sets of rectilinear paths. To avoid potential confusion between the distance traveled and path length, I contextualized both tasks as comparing the lengths of "strings" or "paths."

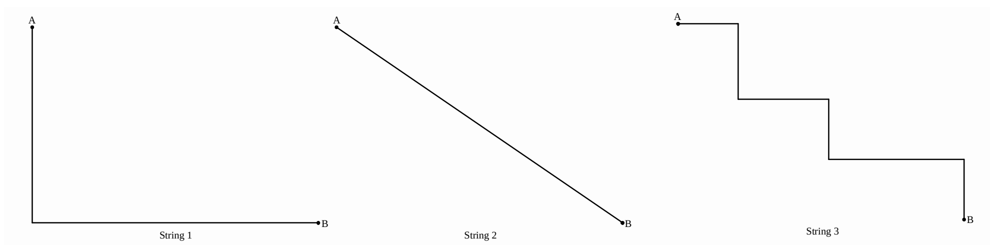


Figure 19. Image of strings shown for Interview 1 Task 1.

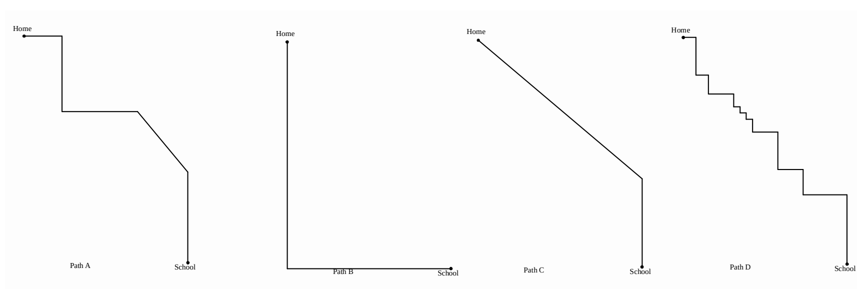


Figure 20. Image of paths shown for Interview 1 Task 2.

The strings for Interview Task 1 and paths for Task 2 (Figures 19 and 20 above) were each printed on a separate transparency page. I overlapped all of the transparencies to show that the strings or paths connected the same points, designated as "A" and "B" for Task 1 and "Home" and "School" for Task 2. I asked students to compare the strings or paths by their lengths. I included a series of pre-planned follow-up questions in the protocol to probe students' intuitive and analytical strategies for defending their claims about the order of the strings or paths by their lengths.

I also included tasks involving comparing a curve and a straight object (Clements et al, in press) to probe students' intuitive and analytical thinking for curves.

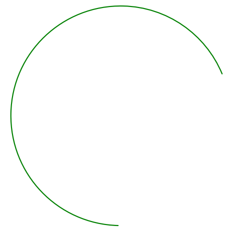


Figure 21. Image of curve shown for Interview 1 Task 3.

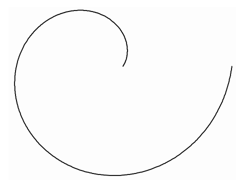


Figure 22. Image of curve shown for Interview 1 Task 4.

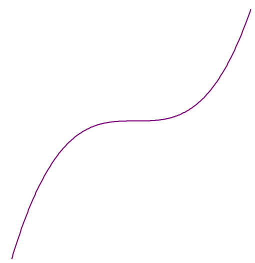


Figure 23. Image of curve shown for Interview 1 Task 5.

For Interview Tasks 3, 4, and 5, I provided students with an image of a curve printed on a standard piece of paper, a 4-in. wooden stick, and a pen. I then asked students to compare the length of the curved path and the stick. I included a series of pre-planned follow-up questions in the interview protocol designed elicit students' explanations about their ways of comparing the curve and the stick, whether they thought they had over- or underestimated when comparing, and why they thought they had over- or underestimated.

Overview of Interview 2 Tasks

The second interview consisted of curvilinear path comparison tasks, Interview Tasks 6A, 7, and 8A, tasks involving comparing two curves using a straight object, Interview Tasks 6B and 8B (Clements et al., in press), and tasks involving measuring a curve with a ruler, Interview Tasks 9 and 10 (Grugnetti, Rizza, & Marchini, 2007).

The purpose of including the three curvilinear path comparison tasks was to extend the literature on path length intuition by probing students' intuitive and analytical thinking for comparing sets of curvilinear paths by their lengths.

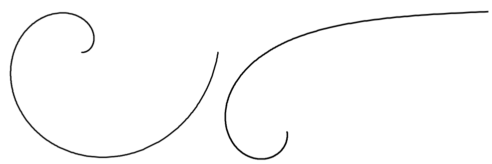


Figure 24. Image of curve shown for Interview 2 Tasks 6A and 6B.

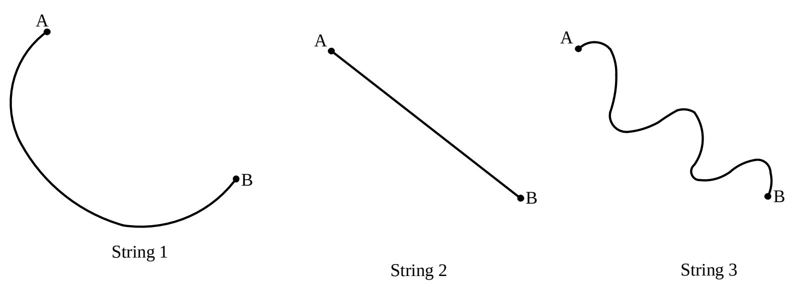


Figure 25. Image of curve shown for Interview 2 Task 7.

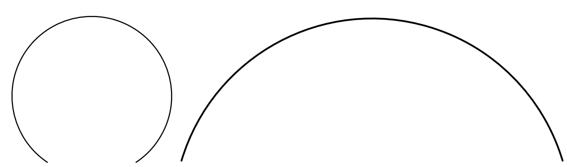


Figure 26. Image of curve shown for Interview 2 Tasks 8A and 8B.

The curves for Tasks 6A and 8A were printed on standard pieces of paper. I asked students to compare the curves by their lengths without tools. For Task 7, each curve was printed on a separate transparency page. I overlapped the transparencies to show that the strings connected the same points, designated as A and B. I included a series of pre-planned follow-up questions for Tasks 6A, 7, and 8A to probe students' intuitive and analytical thinking while defending their claims about their order of the curves.

I included Tasks 6B and 8B to further probe students' intuitive and analytical thinking for curves. After comparing the two curves without tools, I gave students a 4-inch stick, which is a nonstandard unit, and a pen. I then asked them to use the stick to help them check the comparison they had made without tools about the order of the curves by their lengths. Similar to the structure of Interview Tasks 3, 4, and 5, I included a series of pre-planned follow-up questions to probe students ways of comparing the curves and straight object, whether they had over- or underestimated when comparing, and why they though they had over- or underestimated.

I selected Interview Tasks 9 and 10 to probe students' intuitive and analytical thinking for curves when using a standard tool for measuring length, a ruler. To vary the representation of the unit as well as to contextualize the image as an outline of a doorway on a blueprint, I printed the curves for Tasks 9 and 10 on gridded paper.

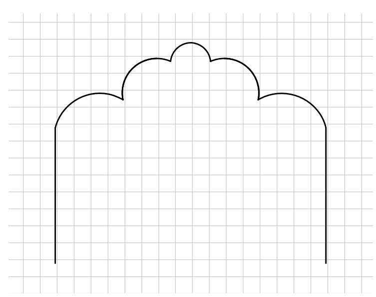


Figure 27. Image of curve shown for Interview 2 Task 9.

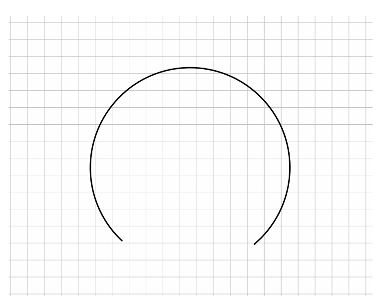


Figure 28. Image of curve shown for Interview 2 Task 10.

I gave students a standard ruler and a pen. I told them that the curve on the paper was the outline of a doorway on a blueprint, and asked them to measure the outline of the doorway in the most precise possible way. I included a series of pre-planned follow-up questions in the protocol for the purpose of probing students' use of intuitions and analytical strategies while measuring the curve using a standard ruler.

Interview Data Analysis

The data in this study were derived from the written length LT-based assessment and structured, task-based interviews. In addition to students' written responses to the written length LT-based assessment, sources of data subjected to analysis included videotaped records of the task-based interviews and transcripts of these interviews, my reflections, and students' written work generated during the interviews. The sections below describe the methods and procedures that I used to analyze the interview data.

I distinguished intuitions and analytical strategies from each other according to the definition and properties of an intuition as outlined by Fischbein (1987). Fischbein defined intuition as "a primary phenomenon which may be described but which is not reducible to more elementary components" (p. ix). Intuitive statements are ones that appear to be immediate, direct, and global. I regarded observable behaviors, including statements, gestures, or manipulations of tools, which did not meet Fischbein's definition and properties of intuitions, as evidence of analytical thinking. Because intuitions and analytical strategies for comparing sets of rectilinear or curvilinear paths are not described in the length LT, I described segments of data in the interviews using a combination of codes from prior research on sixth grade students' intuitions about path length (Chiu, 1996), and emergent codes generated through a constant comparative method of analysis (Merriam, 2009).

I defined a unit to be the smallest meaningful segment of data within each task for each participant. For the purpose of the analysis of this study, a segment of data had to meet two criteria to be considered a unit (Lincoln & Guba, 1985). First, the segment must reveal information relevant to the study, which means that the segment must reveal the student's intuitive or analytical thinking. Second, the segment must be "the smallest piece of information about something that can stand by itself" (p. 345).

I selected each task to evoke one or more units. For example, on Interview Task 2 involving comparing a set of rectilinear paths by length with the follow-up question, "Why is Path B the shortest?" a student might have said, "This path is the shortest because a straight line is the shortest path between two points." This response reflects a unit and would be assigned one code. For another follow-up question, "Why is Path D the longest?" the same student might have said, "This path is the longest because it has a lot of turns." This response reflects another unit and would have been assigned another code. The sections below the code development and frequency analysis.

Code Development Process

After I conducted the interviews I transcribed them, including descriptions of students' gestures and ways of using the tools while engaging in each of the ten interview tasks. I reviewed the transcripts and my post-interview reflections and, for each task for each participant, I identified relevant units of data. Through an initial cycle of open coding (Corbin & Strauss, 2008), I made comparisons among units of data among participants, and developed codes to identify qualitatively different instances of intuitive and analytical thinking that reoccurred with regularity.

I then extracted thematic categories from the list of initial codes through axial coding (Corbin & Strauss, 2008). I constructed these thematic categories to be (a) representative of what is in the data, (b) exhaustive, (c) mutually exclusive, and (d) conceptually congruent (Merriam, 2009). For example, I categorized codes as either pertaining to an intuition or an analytical strategy. Within each of these broad categories of intuitive and analytical thinking, I developed codes to describe and differentiate different types of intuitive and analytical thinking for four groups of similar tasks (see Table 4 above): comparing sets of rectilinear paths by their lengths (Tasks 1 and 2), sets of curvilinear paths by their lengths (Tasks 6A, 7, and 8A), curves and a straight object (Tasks 3, 4, 5, 6B, and 8B), and measuring a curve with a standard ruler (Tasks 9 and 10).

This process yielded a total of 39 codes that I used to describe participants' statements, gestures, or tool manipulations. I grouped codes into four thematic categories: intuitions, with eight codes; analytical strategies, with 23 codes; analytical strategies with embedded intuitions, with two codes; and descriptors for students' reflections on error, with six codes. After I developed codes inductively in this initial round of open and axial coding for each of the four types of tasks, I deductively applied the coding scheme to all units of data in a second round of coding. See Appendix D for a comprehensive list of the codes, organized by thematic category.

Frequency Analysis

I simultaneously reviewed video records and transcripts to identify units in all 32 interviews. I then color-coded and labeled transcripts using the coding scheme I developed for the study. Figure 29 shows the color coding and labeling of part of the Interview 1 transcript for Task 2 for one participant, Mia.

M: And this one's probably longer cuz it has a lot of straight lines (looking at path D) going up and down.	Complexity intuition
I: OK.	
M: And this one's the longest (looking at path B) cuz it goes down...it has to go all the way down to the bottom of the page and then over there.	Detour intuition
I: Alright. So, let me...let me see...Tell me again why is path C the shortest?	
M: cuz...um...it just has two straight lines (tracing over the segments of path C)	Straightness intuition
I: OK. So, what is it about path C that makes you think it's the shortest?	
M: cuz if these were both straight then it would just be about that long (pointing finger to place on the table where path would end if she were to straighten it).	Compression intuition

Figure 29. Illustration of color coding and labeling of relevant units of data.

After I coded and labeled each of the 32 interview transcripts, I then subjected the coded data to a frequency analysis. I tracked the frequency of each code per student and per task using spreadsheet software. Figure 30 illustrates the tracking of the codes assigned to the relevant units of data for each interview participant for Task 2.

	Task 2									
	Straightness Intuition	Complexity Intuition	Detour Intuition	Compression Intuition	Conflicting Intuitions	Combination of Intuitions	Analytical Strategy Use	Total Number of Intuitions Used	Rejected an Intuition	Used a Rejected Intuition
Mia	2	6	2	2	1	1	0	12	1	2
Kevin	1	2	0	0	0	0	0	3	0	0
Noah	0	3	2	0	0	1	0	5	0	0
Jenny	1	2	1	0	0	0	0	4	0	0
Ned	2	5	4	0	0	4	0	11	0	0
Rose	2	4	0	1	0	1	0	7	0	0
Trent	0	3	0	0	0	0	2	3	0	0
Lynn	0	0	0	0	0	0	2	0	0	0
Grant	2	2	1	0	0	1	0	5	0	0
Rick	1	1	2	0	0	1	0	4	0	0
David	1	2	0	0	0	0	0	3	0	0
Ruth	0	0	0	0	0	0	5	0	0	0
Zane	1	0	0	0	0	0	2	1	0	0
Scott	4	1	2	1	1	2	3	8	1	0
Marie	1	2	3	2	2	2	0	8	0	2
Kyle	0	2	0	5	0	1	0	7	0	0
Totals	18	35	17	11	4	14	14	81	2	4

Figure 30. Example of tracking coded data for each interview participant for Task 2.

Next, I tracked the frequency of each code for each participant across the four groups of similar tasks (see Table 4): comparing sets of rectilinear paths with no tools (Tasks 1 and 2), comparing sets of curvilinear paths with no tools (Tasks 6A, 7, and 8A), comparing curves and straight objects (Tasks 3, 4, 5, 6B, and 8B), and measuring curves with a standard ruler (Tasks 9 and 10). Figure 31 below illustrates the tracking of the frequency of the codes for each participant within a group of similar tasks (Tasks 1 and 2).

	Tasks 1 & 2									
	Straightness Intuition	Complexity Intuition	Detour Intuition	Compression Intuition	Conflicting Intuitions	Combination of Intuitions	Analytical Strategy Use	Total Number of Intuitions Used	Rejected an Intuition	Used a Rejected Intuition
Mia	5	6	4	6	2	2	0	21	2	4
Kevin	2	4	0	0	0	1	0	6	0	0
Noah	2	6	4	0	0	3	0	12	0	0
Jenny	3	3	2	0	0	0	0	8	0	0
Ned	3	6	5	0	0	4	0	14	0	0
Rose	3	5	1	1	0	2	1	10	0	0
Trent	1	4	0	1	0	0	5	6	0	0
Lynn	0	0	0	0	0	0	4	0	0	0
Grant	5	3	2	0	0	2	0	10	0	0
Rick	4	3	4	0	0	4	1	11	0	0
David	2	4	0	0	0	0	0	6	0	0
Ruth	1	0	1	0	0	1	6	2	0	0
Zane	1	0	0	0	0	0	3	1	0	0
Scott	5	3	2	1	1	3	3	11	1	0
Marie	4	3	5	3	3	3	0	15	1	2
Kyle	1	2	1	6	0	2	0	10	0	0
Totals	42	52	31	18	6	27	23	143	4	6

Figure 31. Example of tracking coded data for each interview participant for a group of similar tasks (Tasks 1 and 2).

After I tracked codes for each participant for groups of similar tasks, I tracked the frequency of each code for groups of students who represented the length LT levels. Figure 32 below illustrates this tracking of coded data within and across a group of participants who represented particular length LT levels for a group of similar tasks (Tasks 1 and 2).

	Tasks 1 & 2									
	Straightness Intuition	Complexity Intuition	Detour Intuition	Compression Intuition	Total Number of Intuitions Used	Combination of Intuitions	Conflicting Intuitions	Rejected an Intuition	Used a Rejected Intuition	Analytical Strategy Use
CLM Group	12	19	10	6	47	6	2	2	4	0
CRM Group	7	15	6	2	30	6	0	0	0	10
ICPM Group	12	10	7	0	29	7	0	0	0	7
ALM Group	11	8	8	10	37	8	4	2	2	6
Totals	42	52	31	18	143	27	6	4	6	23

Figure 32. Example of tracking coded data within and across participants representing specific length LT levels.

Finally, I examined developmental patterns across these groups of participants who represented particular length LT levels for each of the four groups of similar tasks (see Table 4). Findings from this frequency analysis informed the elaboration of the four levels of the length LT that I addressed in this study: the CLM, CRM, ICPM, and ALM levels.

CHAPTER IV

RESULTS AND DISCUSSSION

This chapter describes results pertaining to length measurement, as well as results related to students' intuitive and analytical strategies for comparing rectilinear and curvilinear paths. In the first section, I characterize participants' level of sophistication for length measurement, as measured by a written assessment based on a hypothetical learning trajectory (LT) for length measurement (Clements et al, in press). Following this are sections in which I (a) describe and differentiate the intuitive and analytical strategies for rectilinear and curvilinear paths that I observed across 16 interview participants, and (b) compare students' responses within and across adjacent LT levels to relate students' intuitive and analytical strategies for rectilinear and curvilinear paths to the LT for length measurement.

Length Measurement

The sections below describe results of the length LT-based assessment for the 16 students who were selected to participate in two individual interviews. Specifically, the following sections illustrate how the length LT was used to analyze students' responses to each of the items on the assessment and categorize them into LT groups that represent four levels: Consistent Length Measurer (CLM), Conceptual Ruler Measurer (CRM), Integrated, Conceptual Path Measurer (ICPM), and Abstract Length Measurer (ALM).

CLM Level Group

The CLM level group represents the lowest level of the length LT included in the individual task-based interviews. This group consists of four Grade 4 students, Jenny, Mia, Noah, and Kevin, who exhibited predominantly CLM-level thinking and provided similar responses on the seven tasks on the written assessment. The following sections include descriptions of their responses to each of the assessment tasks, and the analysis of these responses using the length LT.

CLM level tasks. Jenny, Mia, and Noah correctly answered both of the broken ruler tasks, Tasks 1 and 2 (see Figures 9 and 10, Chapter 3). Their ability to answer both of these tasks correctly, including the task involving fractions, suggests that they could see a ruler as a collection of iterated units and understand the zero point on the ruler, which are concepts that are consistent with the CLM level of the length LT. Therefore, their responses on these tasks suggest that they are operating at least at the CLM level.

Kevin incorrectly answered "6 in" for Task 1. This suggests that he may have been counting tick marks, which is an LURR level strategy. He then correctly answered "3 ½ in" for Task 2. Kevin's correct numerical response on Task 2 indicates that he was at least beginning to develop the CLM concepts of seeing a ruler as a collection of iterated units and understanding the zero point on the ruler.

CRM level tasks. All of the students in the CLM level group, Jenny, Noah, Mia, and Kevin, gave incorrect responses to both of the CRM level tasks, Tasks 3 and 4 (see Figures 11 and 12, Chapter 3). For Task 3, Jenny answered 10 cm Kevin answered 6 cm, Mia answered 7 cm, and Noah answered 11 cm. Each of these responses is only 2 cm off from the correct answer of 9 cm for the length of the missing side; however, none of the students used the CRM level strategy of projecting or translating a given length to determine a missing length.

On Task 4, none of the students in the CLM level group provided a correct numerical response of 210; Jenny incorrectly answered 200, Noah and Kevin both answered 180, and Mia answered 150. Kevin included some addition, written vertically along the side: "60 + 40 + 20 + 30 + 30 = 180." This suggests that he added the labeled segments (60, 20, 40, and 30), and estimated the length of only one of the unlabeled segments as 30. Mia included the calculation 20 + 40 + 30 + 60 = 150, written vertically. This indicates that she added only the labeled segments of the path on Task 4 and did not attend to the missing measures. Jenny, Kevin, Mia, and Noah's incorrect responses to Tasks 3 and 4 suggest that they did not project or translate the given lengths in the diagrams to determine the missing lengths, which is a strategy that children who have developed CRM level concepts and processes would apply to these tasks. Therefore, their responses on Tasks 3 and 4 indicate that they are not yet at the CRM level.

ICPM and ALM level tasks. Jenny, Kevin, and Noah gave similar responses for the ICPM and ALM level tasks (see Figures 13, 14, and 15, Chapter 3). For Task 5a, which asked how many different L-shaped paths they could make with a string that is 10 cm long, and 5b, which asked them to explain how they got their answer and why they think it is correct. Jenny answered five and wrote; "I (tried to) made each turn one cm long, so if it takes two turns for one L-shape then I have five. ($10 \div 2 = 5$)." Kevin answered eight and drew eight L-shaped paths, each with the side lengths labeled as five. He wrote, "You have to flip and swith [sic] to get them." Noah answered 4 and explained how he got his answer by writing, "you could form a square." Jenny, Kevin, and Noah's responses indicate that they were not able to anticipate and monitor sets of related cases of L-shaped bent paths, which involves mental actions that are consistent with ICPM level thinking. This suggests that either they misunderstood the question, or they do not yet possess ICPM level concepts and processes.

Within this group, Mia provided a unique response. On Tasks 5a and 5b, she wrote that she could form “15” L-shaped paths with the string, “because I keped [sic] making one side shorter and the other longer and I made 15 that could worked [sic] and then I ran out of string.” She included the following drawing of 15 L-shaped paths in a line across the page (Figure 33)

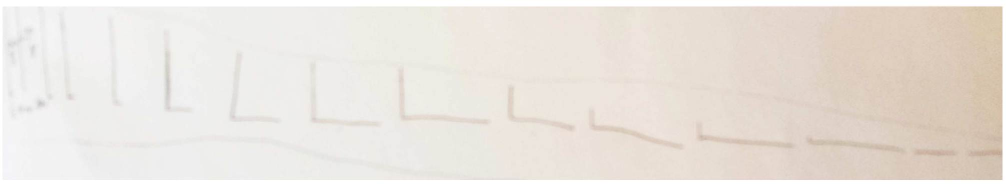


Figure 33. Mia’s set of 15 L-shaped paths made with 10 cm of string.

The first path on the left had a tall vertical side and a short horizontal side. As Mia drew the paths across the paper, the vertical side became shorter and the horizontal side became longer until the final path was nearly a horizontal line. She labeled the vertical sides of the two leftmost paths as 9 cm and 9 ½ cm. She labeled the horizontal side of one of these paths as ½ cm; however, it was not clear for which path this label was intended. She did not label the side lengths of any of the other L-shaped paths she had drawn. On this task, Mia showed that she was able to think about, at least in a qualitative way, coordinating a series of changes in a systematic way across multiple figures. When this coordination also involves the association of space and number, it is consistent with the ALM level of the length LT. Therefore, Mia’s response on this task suggests that she may be developing ICPM and ALM concepts.

On Task 6a, Noah (Figure 34) and Kevin (Figure 35) gave similar responses. Although their rectangles did not reflect geometric inconsistencies, neither drew rectangles with a perimeter of 2 inches.

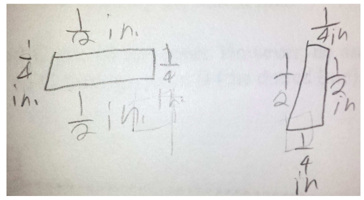


Figure 34. Noah's rectangles.

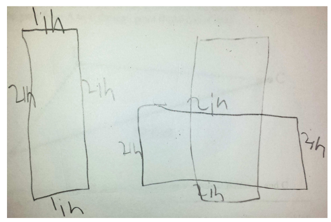


Figure 35. Kevin's rectangles.

Noah and Kevin's responses suggest that they are not yet able to determine side lengths from perimeter, at least when the situation requires them to fracture the unit. This indicates that they have not developed the ability to accurately operate on multiple units and collections of units or on subunits, which is ICPM level thinking.

Jenny (Figure 36) and Mia (Figure 37) exhibited similar ways of thinking on Task 6a. Both Jenny and Mia drew figures that had a perimeter of 2 inches.

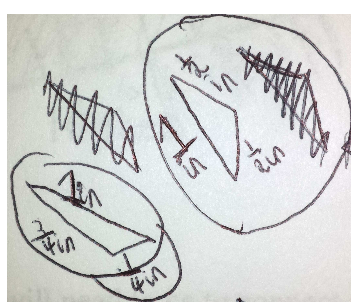


Figure 36. Jenny's triangles.

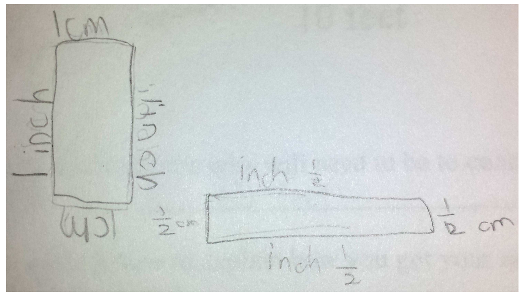


Figure 37. Mia’s rectangles.

Rather than sketching rectangles, Jenny sketched two different triangles that both had a perimeter of 2 inches. She labeled the side lengths as 1 in., ¾ in., and ¼ in. for one triangle and 1 in., ½ in., and ½ in. for the second triangle. In Mia’s sketch, one rectangle had a perimeter of 2 inches (the ½ inch by ½ inch rectangle), and another had a perimeter of approximately 2 inches (the 1 cm by 1 inch rectangle). Although both of Jenny’s triangles violate the triangle inequality, her and Mia’s responses here show that they could think about determining side lengths from perimeter even in the case when the situation requires her to fracture the unit. This suggests that they may be developing the ability to operate on multiple units and collections of units or on subunits, which is ICPM level thinking.

For Task 6b, all four students provided similar responses. Jenny answered three, Kevin answered zero, Mia answered five, and Noah answered that two more rectangles would have a perimeter of 2 inches. This suggests that none of the students in this group have developed a continuous sense of length, which develops later at the ALM level of the length LT.

All of the students in the CLM level group provided correct responses for Tasks 7a, b, c, and d by not violating the triangle inequality. Kevin, Mia, and Noah’s written work indicated that they used similar strategies for these tasks. In Task 7a, when asked how long the wire would need to be to connect points A and C through B, Kevin answered “14 feet” without making any marks on the diagram or showing any work on the page. On Task 7b, he explained how he got his answer by drawing a triangle with the vertices labeled as A, B, and C like the one provided on the page. He labeled the segment from A to C as 10 and the segment from B to C as 21. His labeling of $\overline{BC}$ as 21 in Task 7b is inconsistent with his response of “14 feet” on Task 7a; however, he provided no explanation about his thinking. Mia answered “12 feet.” On the diagram, she labeled $\overline{AB}$ as five and $\overline{BC}$ as seven. She also drew a segment connecting point B with approximately the middle of $\overline{AC}$. For Task 7b, she explained “I put my fingers on the B and moved them to the 10 feet line it was in the middle or 5 so that was that anser [sic]”. Then I put my fingers on the five and moved them to the other line 7.” Noah answered “25 ft.” without writing any calculations or making any marks on the diagram. For Task 7b, he wrote “B + C = 15” without offering an explanation of why he thought the sum of the lengths of these segments should be 15.

On Task 7c, when asked how much wire he would buy so that he could be sure to have enough to connect points A and C through B, Kevin answered that he would buy “20 feet.” He explained his thinking on Task 7d writing, “It is beter [sic] to have more in case you mis mesuer [sic] or brake some.” Mia explained that she would buy “15 feet” of wire because “I think there should be extra in case I got the wrong number.” Noah explained that he would buy “30 ft.” of wire because “A + B + C = 30 ft.”

Kevin, Mia, and Noah’s responses to the parts of Task 7 are plausible. That is, they did not violate the triangle inequality. However, the context of the problem or the inclusion of the diagram on the page may have helped them answer correctly without engaging the concepts and processes described in the levels of the length LT. Therefore, I made no level claim for them for Task 7.

Jenny (Figure 38) provided a unique response within this group for Task 7. On Task 7a, Jenny answered “12 ½ ft.”

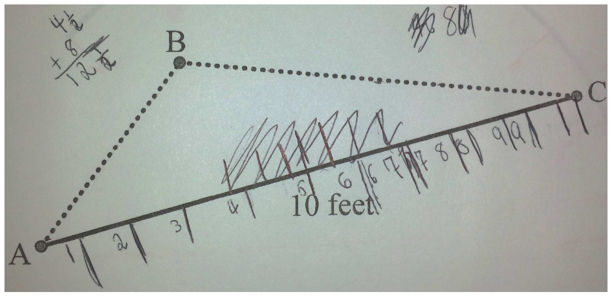


Figure 38. Jenny’s partitioning.

She made tick marks on $\overline{AC}$ to partition $\overline{AC}$ into 10 segments. She labeled $\overline{BC}$ as 8, but she did not make tick marks on either $\overline{BC}$ or $\overline{AB}$. Jenny vertically wrote 4 ½ + 8 ½ = 12 ½, presumably because she thought that $\overline{AB}$ was 4 ½.

For Task 7b, she explained, “A to C was ten ft, so I found out how long one foot is. Then I used it on A to B to C.” This suggests that Jenny partitioned $\overline{AC}$ to find a unit that she could iterate, either physically with her fingers or mentally on $\overline{BC}$ and $\overline{AB}$, to determine the length of the bent path from A to C through point B. For Task 7c, Jenny said that she would buy “14 ft” of wire to be sure that she had enough to connect points A and C through B. On 7C, she explained “I got 14 ft so I would have 1 ½ inches extra.” Jenny’s ability to partition a 10-unit segment into 10 same-size pieces to create a unit and then operate on that unit to measure is evidence that she may be developing an internal measurement tool, which is consistent with the CRM level of the length LT.

CLM level group summary. Overall, on this assessment, Jenny, Kevin, Mia, and Noah all showed evidence that they were operating predominantly at the CLM level of the length LT. Kevin showed that he was still reverting back to LURR level strategies by counting tick marks on one of the broken ruler tasks. Two students, Jenny and Mia showed evidence that they were developing some of the concepts and processes consistent with higher levels of the length LT than the CLM level. Specifically, Jenny's responses indicated that she might have been beginning to develop some of the concepts and processes at the CRM level, and Mia showed evidence of ICPM level thinking.

CRM Level Group

The CRM level group consists of three Grade 6 students, Trent, Ned, and Rose, and one Grade 8 student, Lynn, who exhibited predominantly CRM-level thinking and responded in similar ways to the set of seven tasks on the written assessment. The sections below describe their responses to the assessment tasks, and the coding of these responses using the levels of the length LT.

CLM level tasks. Ned and Trent both provided the same correct numerical answers to the CLM level tasks, Tasks 1 and 2. They both correctly answered "5 in," for Task 1 and "3 ½ in." for Task 2. Their ability to correctly resolve these broken ruler tasks, indicates that they have an understanding of the zero point and see a ruler as a collection of iterated units. This suggests that both Ned and Trent were operating *at least* at the CLM level of the length LT.

Rose and Lynn responded in similar ways to these broken ruler tasks. They both incorrectly answered "6 in." for Task 1. Lynn explained her answer by writing, "Because the one on the rule is not shown. Therefore you are going to subtract 1 inch from your answer." This suggests that they counted tick marks or have a misconception about the zero point on the ruler, which consistent with the LURR level of the length LT. Lynn went on to correctly answer “3 ½ inches” for the second broken ruler task; however, Rose incorrectly answered “4 in” for Task 2. Rose, drew loops to connect the numbered tick marks on the image of the broken ruler and drew a circle around the interval between the tick mark corresponding to 6 ¾ and the tick mark labeled as 7. This suggests that both Lynn and Rose are at least beginning to develop the CLM level concepts of understanding the zero point on the ruler and seeing the ruler as a collection of iterated units. Therefore, Lynn and Rose’s performance on Tasks 1 and 2 suggest that they were operating within the LURR and CLM levels of the length LT.

CRM level tasks. For Task 3, Ned, Rose, Trent, and Lynn all correctly answered “9 cm.” Ned and Rose, included no markings on the page, but Trent and Lynn also included the calculation 22 – 13 = 9, written vertically. Although all of the students in this group correctly answered Task 3, all of them gave incorrect responses for Task 4. On this task, Ned, Rose, and Lynn all answered 150. Ned included no calculations, but Rose and Lynn each included a calculation on the side of her paper, which indicated that they added only the labeled segments of the path and did not attend to the missing measures (For example, Rose included the calculations: 60 + 20 = 80, 40 + 30 = 70, and 80 + 70 = 150, written vertically). Trent incorrectly answered “220” and included no calculations. Therefore, it is not clear whether he applied that same strategy to Tasks 3 and 4, making computational error on Task 4, or if he estimated the lengths of the unlabeled segments.

Ned, Rose, Trent, and Lynn’s inconsistent responses on Tasks 3 and 4 suggest that they can project or translate given lengths to determine missing lengths in some situations (such as Task 3), which is a strategy that a child who has developed CRM level concepts and processes would apply to these tasks. Therefore, their responses on Tasks 3 and 4 indicate that they were beginning to develop CRM level thinking.

ICPM and ALM level tasks. On Task 5, Ned, Rose, Lynn, and Trent exhibited similar ways of thinking. Ned answered that he could form two different L-shaped paths from a string that is 10 cm long. He defended his answer by writing "I think 2 because it may require a lot of bending the string." Rose responded that she could form four and explained how she got her answer writing, "I don't know what the question is asking so I thought of turning the 'L' so I turn it (Rose drew an L-shaped path and then three additional versions of it rotated at 90, 180, and 270 degrees) and I got 4 paths." This suggests that Rose only attended to the orientation of the L-shaped path and did not attend to creating L-shaped paths of varying side lengths. Lynn wrote that she could form five, "Because there is only so much space for so little of Ls. If this is the L. You can form them out or down." She drew a single L-shaped path with two horizontal rays projecting out of the vertical segment and two more vertical rays projecting out of the horizontal segment. Trent drew a single bent path with six segments, labeling the segments of the path as 1 cm, 2, cm, 1 cm, 4 cm, 1 cm, and 1 cm. He also wrote that he would be able to form five and explained his thinking by writing, "I think my answer is correct because if you add 1 + 2 + 1 + 4 + 1 + 1 you would get 10 and I made 5 L's". Ned, Rose, Lynn, and Trent's responses to Tasks 5a and 5b suggest that they misunderstood the question, or they did not yet possess the concepts and processes at the ICPM level of the length LT.

On Task 6a, Ned, Lynn, Trent, and Rose all provided similar responses. Ned and Lynn both drew two squares, both with all four sides labeled as ½ in. Rose drew a ¾ in by ¼ in rectangle and a ½ in by ¾ in rectangle. Trent drew two rectangles, both with perpendicular adjacent sides labeled as ¾ in by ¼ in. There were no geometric inconsistencies in their sketches, but each student produced only one rectangle that had a perimeter of 2 inches. Therefore, Ned, Lynn, Trent, and Rose are able to think about determining side lengths from perimeter even in the case when the situation requires them to fracture the unit. This suggests that they may be developing the ability to operate on multiple units and collections of units or on subunits, which is consistent with ICPM level thinking.

For Task 6b, Ned and Rose both said three more rectangles would have a perimeter of 2 inches. Lynn said there would be only one more, and Trent answered that there would be zero more. This indicates that Ned, Rose, Lynn, and Trent do not yet have a continuous sense of length, which develops later at the ALM level of the length LT.

For Task 7, Ned, Rose, Trent and Lynn all provided similar correct responses; they did not violate the triangle inequality. Both Ned and Rose said that it might take 13 feet of wire to connect points A and C through B. Trent answered 12 feet, and Lynn answered 14 feet. Ned and Trent both wrote about estimating when explaining their thinking for Task 7b. Ned wrote, "I used my fingers to make a path to the line AC to guess how long that would be then I added both numbers." Trent defended his answer of 12 feet by writing, "Because if I estimated correctly $\overline{BC}$ should be about 8 ft and $\overline{AB}$ should be about 4 ft and 8 + 4 = 12."

Rose and Lynn gave similar explanations of how they got their answers in Task 7b. Rose explained, "I got my answer from using my fingers to go from point to point then compared it to the line that is 'ten ft' and guessed how long, added the dotted lines and got my answer. I think it's correct because it seems reasonable." Lynn wrote that she thought the wire would need to be 14 feet long because "If you take your fingers and go from point A to point C it's 10 ft. If you keep your fingers that far apart and go a little over point B, that'll be 10 ft. It looks to me that from the top of point B to the point of C is 9 ft. from A to the top of B is 5 ft. If you take your fingers and keep that and connect it with the 10 ft, it's 5 (half-way) 9 + 5 = 14." Presumably, Lynn and Rose each measured the distance between points A and B by spanning the gap from her thumb to her index finger and checked to see if that span fit the gap from points A to C and A to B. Therefore, both Lynn and Rose made an indirect comparison between each of $\overline{AB}$ and $\overline{BC}$ to $\overline{AC}$.

On Task 7c, Both Ned and Rose explained that they would buy 15 feet of wire to make sure they would have enough to connect points A and C through B. Ned defended his answer by explaining, "I think this is correct because because [sic] I'm making sure I have enough wire to get from A to C through B" on Task 7d. Rose explained her thinking by writing, "How I got it is I rounded 13 and why I think it's correct is because you could cut the wire to 13 feet if you had 14 and if maybe it was 13.5 you needed, you would have it."

Trent and Lynn provided responses that suggested they interpreted the question to mean that they would need enough wire to connect points A and C through B, and then back from point C to point A again, forming the entire triangle with wire. Trent explained that he would buy "25 feet of wire and defended his answer by explaining, "I think my answer is correct Because 10 + 12 = 22 and if you want to be sure you have enough wire I'd at most get 25 ft of wire." Lynn explained that she would buy 30 feet of wire because

“If point B is also 10 ft then we’ll need an extra 10 ft. But we don’t know so if point B is over 20 ft I'll have enough instead of being short a couple feet of wire.”

Ned, Rose, Trent, and Lynn provided plausible responses to all of the parts of Task 7. Meaning they did not violate the triangle inequality and they all talked about buying extra wire to make sure they would have more than enough to connect points A and B through C. However, like the students in the CLM level group, their articulation of their thinking does reflect the concepts and processes that are described in the levels of the length LT. Therefore, I made no level claim for them for Task 7.

CRM level group summary. Overall, on this assessment, Lynn and Rose’s responses indicate that they are still falling back to use LURR level and CLM level thinking on broken ruler tasks. Ned and Trent’s performance on this assessment indicate that they are still reaching back to use CLM level thinking on tasks in which the level is relevant. However, all four students also showed that they could operate predominantly using CRM level strategies on tasks that require CRM level thinking. In addition, they all showed that they might be beginning to develop some of the concepts and processes at the ICPM level. Therefore, the level that best characterizes the concepts and processes that Lynn, Rose, Ned, and Trent exhibited on the written length LT-based assessment is the CRM level.

ICPM Level Group

The ICPM level group is comprised of one Grade 6 student, Grant, and three Grade 8 students, David, Rick, and Ruth. All of the ICPM level students used predominantly ICPM level strategies and provided similar responses to the seven tasks on the written assessment. The following sections include descriptions of their responses and the coding of these responses using the length LT levels.

CLM level tasks. All of the students in the ICPM level group, Grant, David, Rick, and Ruth provided similar responses to the broken ruler tasks, Tasks 1 and 2. They all correctly answered "5 inches" for Task 1. Grant, David, and Rick included no markings on the page, but Ruth included the calculation 7 – 2 = 5, written vertically. For Task 2, Grant and David correctly answered and 3 ½ inches; however, Rick and Ruth both provided incorrect responses of 3 ¼ inches. Rick did not include any work or explanation, but Ruth included the calculation 6 ¾ - 3 ¼ = 3 ¼, again written vertically. Rick and Ruth's incorrect answers both reflect a likely computational error, rather than a misconception about the ruler. Therefore, Grant, David, Rick, and Ruth's responses to the Tasks 1 and 2 indicate that they see a ruler as a collection of iterated units and have an understanding of the zero point on the ruler, which are both CLM level concepts. Therefore, their responses to these tasks provide evidence that they were operating *at least* at the CLM level.

CRM level tasks. All four students in the ICPM level group correctly answered both of the tasks designed to elicit CRM level thinking, Tasks 3 and 4. On Task 3, Grant, David, Rick, and Ruth all answered 9 cm. Grant, David, and Rick did not make any markings on the page, but Ruth included the calculation 22 – 13 = 9, written vertically. For Task 4, all four students answered 210. Again, Grant and Rick did not show any work on the page or offer any explanation of their thinking. David included the calculation 60 + 20 + 40 + 30 + 60 = 210 and Ruth wrote 120 + 90 = 210; both calculations were written vertically. Grant, David, Rick, and Ruth's responses on these tasks indicate that they can project or translate the given lengths to determine missing lengths, which is a strategy that a child who is at least at the CRM level would use to solve these tasks. Therefore, their responses on Tasks 3 and 4 indicate that they could use CRM level strategies to solve tasks for which the level is pertinent.

ICPM and ALM level tasks. On Task 5, Grant, David, and Ruth all provided similar responses. Grant and David both answered that they could form five L-shaped paths with the 10-cm string. Grant wrote, "You could have an L by having a string that is 9 cm + 1 cm, or 8 cm + 2 cm, or 7 cm + 3 cm, or 6 cm + 4 cm, or 5 cm + 5 cm." Along the side of the paper, David wrote five pairs of numbers: 1 and 9, 2 and 8, 3 and 7, 4 and 6, and 5 and 5, and he explained his thinking by writing "I think it would be correct because there are five ways you can make ten and even if you put it facing a different way it would still be the same." Ruth (Figure 39) answered that he could form 40 paths, and she drew a set of ten paths in a line from left to right.

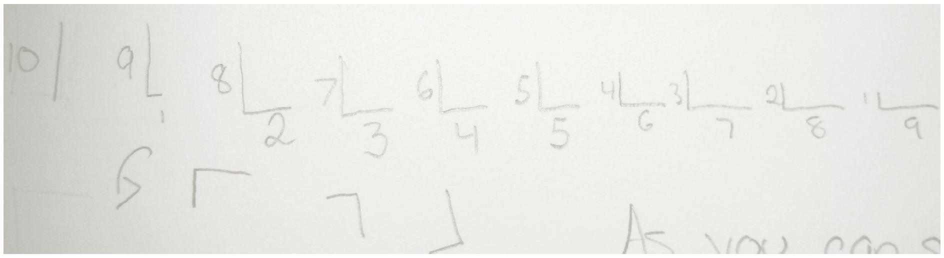


Figure 39. Ruth's set of related L-shaped paths.

The leftmost path was a vertical line, which she labeled as 10. Next, she drew an L-shaped path with side lengths labeled as 9 and 1. This was followed by eight more L-shaped paths with side lengths labeled as 8 and 2, 7 and 3, 6 and 4, 5 and 5, 4 and 6, 3 and 7, 2 and 8, and then 1 and 9. She explained her thinking by writing "As you can see above, there can be different lengths for each side making 9. Also, how about the lower case '1.' You could also do things from different angles (here she drew rotated L-shaped paths). If you do all this, you can get up to 40 different paths." On this task, Grant, David, and Ruth found several related cases of bent paths with the same length and relate those cases to one another to provide evidence that they thought about an underlying pattern. This thinking is consistent with the ICPM level of the length LT; therefore, their responses to this task suggest that they were operating at the ICPM level.

Rick's response to Task 5 was unique within the ICPM level group. He answered that he could form an infinite number of different L-shaped paths, and he explained his thinking by writing, "The long side could be 5.00000…1 cm or something like that. It could be a million 0's." Rick did not show evidence of reasoning about several related cases of bent paths with the same length and relating those cases to one another to provide evidence that he thought about an underlying pattern here, which is ICPM level thinking. However, he showed a growing awareness of a potential infinite number of cases and a continuous sense of length, which develops at the ALM level of the length LT. Therefore, Rick's response to Task 5 suggests that he could have been operating at least at the ICPM level.

On Task 6a, Grant, David, Rick, and Ruth all provided similar responses. Grant drew two triangles that both had a perimeter of 2 inches: one with side lengths labeled as ½ in, ½ in, and 1 in and another with side lengths labeled as ¾ in by ¾ in by ½ in. David drew a 0.6 by 0.4 rectangle and a square with all four sides labeled as 0.5. Rick drew a 0.75 by 0.25 rectangle and a 0.1 in by 0.9 in rectangle. Ruth drew a square with all four sides labeled as ½ and a ¾ by ¼ rectangle. Grant drew triangles rather than rectangles; however, he, along with the other three students in the ICPM level group, drew figures that had a perimeter of two inches. Furthermore, other than one of Grant's triangles, which violated the triangle inequality, none of the student's sketches had geometric inconsistencies. Their responses to this task suggests that they were able to think about determining side lengths from perimeter even in this situation that required them to operate on fractional units. This indicates that Grant, David, Rick, and Ruth had the ability to operate on multiple units and collections of units or on subunits, which is ICPM level thinking.

On Task 6b, when asked David answered that three more rectangles would have a perimeter of two inches. This suggests that David did not yet developed a continuous sense of length, which appears later at the ALM level of the length LT. Grant did not provide a response to Task 6b.

Rick and Ruth provided similar responses to Task 6b. Rick answered that there would be an "infinite" amount of more rectangles that would have a perimeter of two inches. Ruth answered "If you do it in fractions like above, the number is pretty much infinite." She also included the expression $\infty - 2$, written vertically. Presumably, she subtracted two from infinity to account for the two rectangles she had drawn as her response to Task 6a. This provides evidence that Rick and Ruth have a continuous sense of length, which is consistent with the ALM level of the length LT. Rick and Ruth's performance on this task suggests that they were capable of using ICPM level strategies, and that they were beginning to develop ALM level thinking.

For Task 7, Grant, David, Rick, and Ruth all exhibited similar ways of thinking; none of these students provided responses to Task 7 that violated the triangle inequality. Grant and Ruth both answered that they would need 15 feet of wire to connect points A and C through B. Grant explained his thinking in Task 7b writing, "You have to add a little bit because you are going a little off the straight path, so I added a bit." Ruth wrote, "If you image [sic] put the two dotted lines together to make one huge line, and put it next to the solid line, it would be bigger. But most likely not twice as big." This suggests that Grant understands, at least intuitively, that a straight line is the shortest distance between two points, and Ruth relied on a strategy of mentally straightening the bent path to compare it to the straight segment.

David and Rick both answered that they would need 14 feet. David explained his thinking when answering Task 7b writing, "because if $ab^2 + bc^2 = ac^2$, so those two lines have to equal 10 feet and $8^2 = 64 + 5^2 = 25$ is 99 so if you add a little bit and then round up you get 14 ft." Rick wrote, "I did pathagren therom [sic] so 8 + 6 = 14." Although David and Rick's responses here show that they did not violate the triangle inequality, their application of the Pythagorean theorem is overgeneralized to a non-right triangle case.

All of the students in the ICPM level group provided similar responses to Tasks 7c and 7d. They all said they would be 20 feet of wire to make sure they would have enough to connect points A and C through B. Grant explained "You want to make sure you have enough so you should buy a little extra." David explained that he would buy 20 "because that way you can have extra if its [sic] longer than you think it is." Rick defended his answer by explaining, "Double the wire will be more than enough," and Ruth justified her answer of 20 "just in case your estimate is not really 20 feet. So, always go more than you think you need to be safe." Although Grant, David, Rick, and Ruth reasonably answered Task 7, their strategies of mentally straightening the bent path, applying an intuition that a straight line is the shortest distance between two points, and overgeneralizing the Pythagorean theorem do not reflect concepts and processes within the levels of the length LT. Therefore, I made no level claim for Task 7.

ICPM level group summary. Throughout this assessment, Grant, David, Rick, and Ruth's responses indicate that they used CLM and CRM level strategies for situations in which those levels were appropriate, Tasks 1 through 4. For tasks designed to elicit thinking at higher LT levels, they all operated predominantly using ICPM level strategies. Therefore, based on this assessment, I placed all four of these students at the ICPM level. Ruth and Rick also showed that they might have been beginning to develop some of the concepts and processes consistent with the ALM level of the length LT.

ALM Level Group

The ALM level group consists of four Grade 10 students: Marie, Kyle, Scott, and Zane. Each of these four students showed consistent evidence of ALM level strategy use and answered in similar ways to the written assessment tasks. The sections below describe their answers and coding of their answers according the length LT.

CLM level tasks. All four students in the ALM level group provided correct numerical responses to the broken ruler tasks, which were designed to elicit thinking at the CLM level, Tasks 1 and 2. Scott, Marie, Kyle, and Zane all correctly answered five inches for Task 1 and 3.5 in for Task 2. Their correct responses to these tasks indicate that each of these students sees a ruler as a collection of iterated units and understands the zero point on the ruler, which is evidence of CLM level thinking. Therefore, Scott, Marie, Kyle, and Zane used CLM level thinking to resolve tasks that required CLM level concepts and processes.

CRM level tasks. All four students in this group also correctly answered both tasks designed to elicit CRM level thinking, Tasks 3 and 4. Kyle, Scott, Marie, and Zane correctly answered 9 cm on Task 3 and 210 on Task 4. Scott, Kyle, and Marie included no explanations or work for Task 3. Zane included the calculation 22 – 13 = 9, written vertically. For Task 4, Scott, Kyle, and Zane wrote no markings on the page, and Marie included the calculations 20 + 40 + 30 = 90 and 90 + 60 + 60 = 210. These four students' correct responses to Tasks 3 and 4 suggest that they could project or translate the given lengths to determine missing lengths, which is a strategy that a CRM level student could apply on these tasks. Therefore, Kyle, Scott, Marie, and Zane's responses to Tasks 3 and 4 indicate that they could use CRM level thinking in contexts in which the level is relevant.

ICPM and ALM level tasks. On Tasks 5a and 5b, Kyle and Scott provided similar responses. Kyle answered that he could form an "infinite" number of L-shaped paths from a string that is 10 cm long. Scott said that he could form "Any number. Infinite." Kyle explained his answer by writing, "There are so many answers for just have the string like so (he drew a picture of one L-shaped path without labeling the lengths of the sides) because if you adjust it by the smallest degree, the length of both sides would be different than before." Kyle's response does not suggest that he was reasoning about several related sets of paths with the same length, which would have provided evidence that he is operating at the ICPM level of the length LT. However, he did exhibit a continuous sense of length, which develops at the ALM level. Therefore, Kyle's response to this task indicates that he may have been developing ALM level concepts and processes.

Scott defended his answer by writing, “You can have any length for each of the sides of the 90° angle..[sic]” He included sketches of three paths (Figure 40).

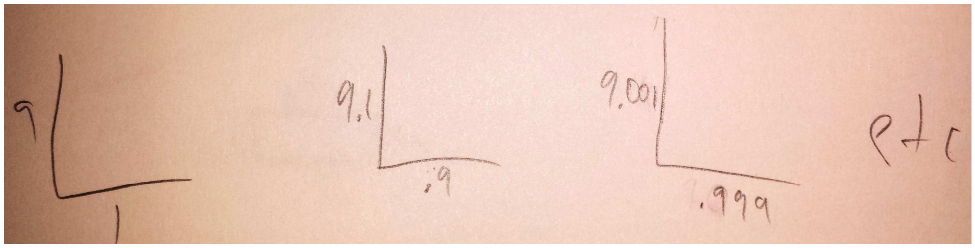


Figure 40. Scott’s sequence of L-shaped paths.

The leftmost path had a vertical segment labeled 9 and a horizontal segment labeled 1. The next path had a vertical segment labeled 9.1 and a horizontal segment labeled .9, and the rightmost path had a vertical segment labeled 9.001 and a horizontal segment labeled .999. Scott’s sequence of three paths suggests that he was reasoning about several related sets of paths here with the same length, and relating those cases to one another to provide evidence that he was thinking of an underlying pattern, which suggests that he is operating at the ICPM level of the length LT. Furthermore, his willingness to suggest that there are infinitely many cases indicates that he has a continuous sense of length, which is evidence of ALM level thinking. Therefore, Scott’s response to this task indicates that had concepts and processes that are consistent with the ICPM and ALM levels.

Marie and Zane provided similar responses to Task 5. On Task 5a, Zane explained that he could form “10 (19 if you count the upside down L’s).” Zane defended his answer of 10 (or 19 when counting the “upside down L’s”) by writing, “Knowing the properties of string I know it probably is really hard to form an L shape that has legs smaller than around 5 mm each. Therefore, one L uses 1 cm of string, but if you count the upside down L’s formed by your regular L’s, you will get 19 full L’s because the upside down ones don’t form a complete L at the end.” Zane’s response to this task suggests that he may have misunderstood the question. Therefore, I made no level claim for him on this task.

Marie answered that he could form "9" different L-shaped paths from a string that is 10 cm long. Along the side of the paper, Marie drew a sequence of nine L-shaped paths. The leftmost path had a short side labeled as 1 cm and a long side labeled as 9. The following paths were similarly labeled as 2 and 8, 3 and 7, 4 and 6, 5 and 5, 6 and 4, 7 and 3, 8 and 2, and 9 and 1. For Task 5b, she explained, "Because you could make an L shape path with 9 cm and 1 cm, 8 cm & 2 cm, 7 cm & 3 cm, 6 cm & 4 cm, 5 cm & 5 cm and then do the reverse." Marie's response to this task suggests that, like Scott in the ALM level group, she was able to find several related cases of bent paths with the same length and relate those cases to one another to indicate that she was thinking about an underlying pattern. This thinking is consistent with the ICPM level.

On Task 6a, Kyle, Scott, Marie, and Zane all gave similar answers. When asked to draw two rectangles that had a perimeter of two inches, Kyle drew a 0.3 by 0.7 inch rectangle and a 0.2 by 0.8 inch rectangle. Scott drew a 0.1 by 0.4 inch rectangle and a square with all four sides labeled 0.25 inches. Marie drew a square with all four sides labeled as ½ in and a ¾ in by ¼ in rectangle. Zane drew a square with all four sides labeled as .5 in and a .75 in by .25 in rectangle. There were no geometric inconsistencies in their sketches, and each of these four students correctly sketched rectangles that had a perimeter of two inches. This indicates that Kyle, Scott, Marie, and Zane could determine side lengths from perimeter, even when the task required them to operate on fractional units. Therefore, they had the ability to operate on multiple units and collections of units or on subunits, which is ICPM level thinking.

On Task 6b, Kyle, Marie, and Zane provided similar answers. Kyle said that an "infinite (the length of the decimals could keep expanding)" number of additional rectangles could be made that would have a perimeter of two inches. Marie answered "an infinite amount if using decimals & fractions." Zane answered "Infinite. For example, one side could go all the way down to the sides of an atom, but the other two sides can still add up to 2 inches." Along the side, he provided the following example: .0000001 + .000001 + .999999 + .999999 = 2 in, written vertically. This indicates that Kyle, Zane, and Marie have a continuous sense of length, which is consistent with the ALM level of the length LT. Their performance on this task suggests that they are capable of using ICPM level strategies, and that they also possess concepts and processes at the ALM level. Scott answered, "Any number." His response to this task is vague and unclear; therefore, I made no level claim for Scott for Task 6b.

For Task 7a, Kyle answered, "about 13 ft." He explained his thinking when answering Task 7b writing, "I moved line $\overline{BC}$ into $\overline{AC}$ and it only appeared to be $\frac{6}{10}$ of line $\overline{AC}$ (6 in). Then I pictured line $\overline{AB}$ coming and connecting to line $\overline{BC}$, which I still have placed inside of line $\overline{AC}$, and the collective length of line $\overline{AB}$ and $\overline{BC}$ appeared to be close to 13 ft." Kyle's response suggests that he mentally straightened the bent path from A to C through B and compared it to the straight path, which he knew was 10 units.

Scott and Marie provided responses for Tasks 7a and b. Like Kyle, Scott also answered, "13 ft.," which he defended by writing, "I estimated (then drew a smiley face)…an educated guess." Marie gave an answer of "about 18 ft. more than 15 but less than 20." She explained her thinking when answering Task 7b writing, "10 will be enough to get from C to B but not enough to get back from B to A and 20 feet would be too long." Scott and Marie's explanations for Task 7b suggest that they relied on estimation to determine their responses for Task 7a.

Kyle, Scott, and Marie responded to 7a with plausible answers for the length of wire needed to connect points A and C through B. However, Kyle's strategy of mentally straightening the bent path to compare its length to the straight segment, and Scott and Marie's strategy of estimating do not reflect concepts and processes within the levels of the length LT. Therefore, I made no level claim for these three students for this task.

Within this ALM level group, Zane used a unique strategy on Task 7. For Task 7a he answered, "11.95 feet," which he explained by writing, "I used my finger to draw a straight line between A to C through B then broke the 10 feet line into fifths (setting two feet) giving me a basic 2 foot estimated measurement to guess my new line" (Figure 41).

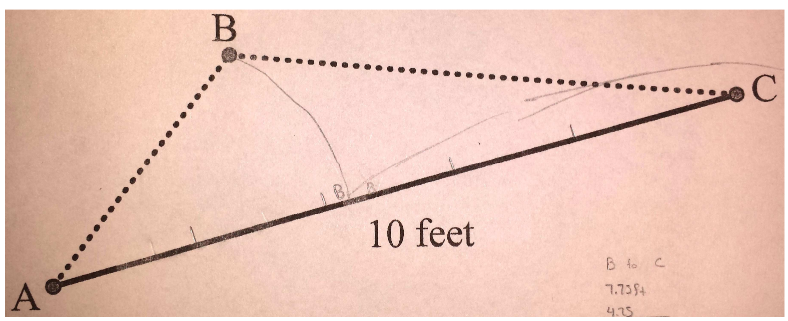


Figure 41. Zane's partitioning.

This suggests that Zane partitioned the segment labeled as 10 to construct a composite unit of 2, which he then operated on, either mentally or physically, to measure the unknown side lengths. Zane's response here indicates that he possessed an internal measurement tool, which develops at the CRM level of the length LT.

Kyle, Scott, Marie, and Zane all answered in similar ways on Tasks 7c and d. On Task 7c, when asked how much wire he would need to connect A and C through B, Kyle answered "14 ft (unless they have a better price for 15 ft)." Marie explained that she would buy "20 ft." Zane explained that he would buy "12.5 ft." Scott said that he would "Buy more than the exact amount (he again drew a smiley face) to be sure…". When answering Task 7d, Kyle defended his answer by explaining, "I want to buy a little bit extra in case my estimation was too short, although I feel that I wasn't off by much." Marie explained, "cause ten feet would cover a little more than enough to get from C to B but not enough to get from B to A." Zane wrote, "It's good to be sure, and I could leave a little room for a guess. To be honest I just put down a number close to my guess." When articulating why he thought his answer was correct, Scott responded, "I still have no idea." Although all of their numerical responses for Task 7c are plausible, meaning they did not violate the triangle inequality, their responses to Tasks 7c and 7d do not reflect the mental actions that characterize the levels of the length LT. Therefore, I made no level claim for the students in the ALM level group for these tasks.

ALM level group summary. Kyle, Scott, Marie, and Zane's responses indicate that they reached back to use CLM and CRM level thinking to resolve tasks pertaining to those levels (Tasks 1 through 4). However, they all also showed that they could operate predominantly using ICPM level strategies on tasks that require ICPM level thinking. Marie showed evidence that she may be beginning to develop some of the concepts and processes at the ALM level. Kyle, Scott, and Zane provided evidence that they could operate predominantly using ICPM or ALM level strategies on tasks that require concepts and processes from the highest levels of the length LT.

Summary of Length LT Groups

Participants in each of the four length LT level groups exhibited the same predominant level of thinking; however, there was still some variability in participants' strategy use within the groups. For example, in the CLM level group, Mia showed evidence that she was beginning to develop concepts and processes that are consistent with the ICPM level, Kevin used LURR level strategies, and Jenny exhibited CRM level thinking. However, all of the students within the CLM level group operated primarily at the CLM level.

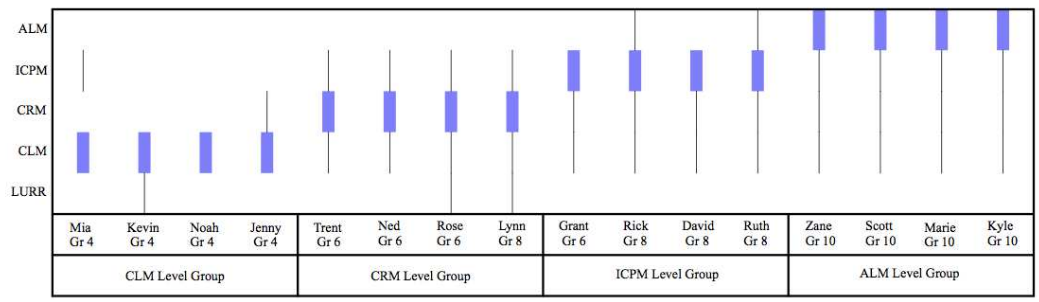


Figure 42. Participant LT level placements.

Figure 42 depicts the variability along with the predominant level observed for each participant within each length LT level group. In Figure 42, a blue rectangle indicates a particular student's main level of thinking, and the thin line indicates the other levels that I observed in the student's work on the written LT-based assessment (Clements et al, in press).

In this section, I established four groups anchored in the LT for length measurement as level representatives for the CLM, CRM, ICPM, and ALM levels (Clements et al., in press). In the following sections, I describe these level representatives' responses to tasks involving aspects of length measurement outside the LT to inform recommendations for extensions to the LT. Specifically, in the sections below I describe and differentiate students' responses, both within and across length LT level groups, to four different categories of conceptually congruent tasks: (a) comparing sets of rectilinear paths by their lengths without tools, (b) comparing sets of curvilinear paths by their lengths without tools, (c) comparing curves and a straight object (a nonstandard unit), and (d) measuring a curve with a standard ruler.

Rectilinear Paths: Intuitions and Analytical Strategies

I posed two tasks (Tasks 1 and 2 in Interview 1) for the purpose of eliciting students' intuitions (Chiu, 1996) and analytical strategies for comparing sets of rectilinear paths. In the next two sections I illustrate the intuitions and analytical strategies that the 16 interview participants used. These are followed by a section in which I describe individual differences with respect to how students used these intuitions and analytical strategies for path length to justify arguments when making comparisons among rectilinear paths. In the final section, I relate intuition and analytical strategy use for path length to the LT for length measurement.

Four Intuitions for Rectilinear Paths

Four qualitatively different intuitions for rectilinear paths were used by the 16 interview participants across Tasks 1 and 2 during the study: straightness, detour, complexity, and compression (Chiu, 1996). Each of these intuitions was identified using Fischbein's (1987) definition of an intuition as "a primary phenomenon which may be described but which is not reducible to more elementary components" (p. ix). A student's statement was considered to be an intuition if it was consistent with properties of intuitions as described by Fischbein. That is, a response was coded as an intuition if it appeared to be an immediate, direct, and global solution to the task. In the following sections, I illustrate how students used each of these intuitions to defend their claims about their ordering of sets of rectilinear "strings" or "paths" by their lengths for Tasks 1 and 2 (see Figures 43 and 44 below), beginning with the straightness intuition.

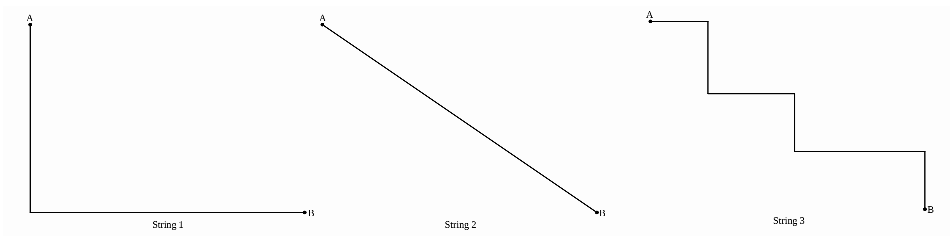


Figure 43. Image of strings shown for interview Task 1.

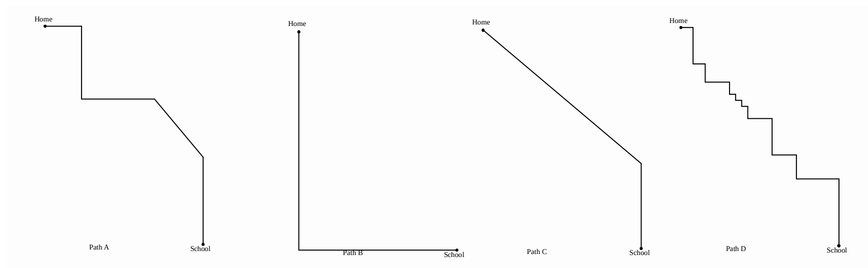


Figure 44. Image of strings shown for interview Task 2.

Straightness. When asked to compare Strings 1, 2, and 3 by their lengths, Jenny (Grade 4, CLM Group) ordered them from shortest to longest as String 2, 3, and then 1. When asked why she thought String 2 was the shortest, she defended her claim using an intuition:

> Interviewer: Can you tell me why you think String 2 is the shortest?
>
> Jenny: um...because it's in a straight line and the other ones...um...are going around longer (traced finger around turns in Strings 1 and 3) so they're longer because they need more string.
>
> Interviewer: OK. What is it about being a straight line that makes it the shortest?
>
> Jenny: Because you...because...cuz it just goes straight and the other ones need more string.

Jenny's response reflects the use of the straightness intuition. It is an intuition because it is an immediate, direct, and global approach to ordering the three strings by their lengths. It is immediate because she provided her response quickly without superimposing the transparencies on which the different strings were printed or attempting to measure first using an improvised tool, such as her finger. Jenny's response is direct and global because it appears to be self-evident to her. Evidence of the global characteristic of the straightness intuition can be derived from Jenny's and the other participants' repeated use of the straightness intuition when defending their orderings of rectilinear "paths" or "strings" by their lengths.

When comparing Paths A, B, C, and D for Task 2 (Figure 44) by their lengths, Ned (Grade 6, CRM Group) used the straightness intuition when describing his ordering saying, "I think Path C is the shortest because it almost goes directly from home to school, but it takes a little bit of a turn and then goes to it." Like Jenny's response, Ned's explanation reflects an immediate, direct, and global approach to ordering the three strings by their lengths. Although none of the paths included in Task 2 were perfect diagonal lines like String 2 for Task 1, Ned used the straightness intuition to defend his selection of the path with the longest diagonal segment as the straightest path from the starting point to the destination.

Detour. Students who used the detour intuition discussed a path as going out of the way or being the least direct. Marie (Grade 10, ALM Group), for example, used the detour intuition to explain why Path B was the longest for Task 2 (Figure 44):

Interviewer: OK. And why is Path B the longest?

Marie: Probably because it goes completely like around (traced along Path B with her fingers) that it might be the longest.

Interviewer: So, what is it about the way Path B looks that makes you think it's the longest?

Marie: It's the least direct.

Marie’s response here is an intuition because it is immediate, direct, and global approach. That is, she answered quickly, it appeared to be self-evident to her, and Marie and the other participants in the study repeatedly used the detour intuition to defend their orderings of the “paths” or “strings” in Tasks 1 and 2 by their lengths.

Complexity. Students who defended their orderings using the complexity intuition discussed the number of a certain feature of the “string” or “path,” such as the number of turns, segments, or angles. For example, Kevin (Grade 4, CLM Group) used the complexity intuition to defend why he thought String 3 was the longest for Task 1 (Figure 43):

Interviewer: Why is String 3 over here the longest?

Kevin: Because it’s like a whole bunch of strings, cuz it’s like do-do-do-do (motioning through the turn with his finger) and it takes up more of the paper.

Interviewer: OK. So, why does having a whole bunch of strings like this...a whole bunch of strings...why does that make a string long?

Kevin: because it's like got all those...like because it's got so many turns, and so it's like so long.

Kevin’s response is an intuition because it is an immediate, direct, and global approach. It was given without further justification or elaboration, and it was repeatedly used in multiple rectilinear path length comparison situations throughout the study.

Compression. Students who used the compression intuition discussed either straightening “strings” or “paths: that were bent or bending “strings” or “paths” that were straight. For example, Kyle (Grade 10, ALM Group) used the compression intuition to defend why String 1 was the longest in his ordering of the strings in Task 1 from shortest to longest as String 2, String, 3 and String 1:

> Interviewer: Why is String 1 the longest?
>
> Kyle: Um...I imagined putting this...the first line A (tracing along the vertical segment of String 1) into the a line looking like String 2, and then adding line B (tracing along the horizontal segment of String 1) to line A, and then it looks...it appears to be longer than String 3. And I did the same thing for String 3.

Rose (Grade 6, CRM Group) also used the compression intuition for Task 2; however, her application of this intuition was different from Kyle's because she used it to defend why she thought Path C was the shortest:

> Interviewer: So, can I ask you why you think this one's (pointed to Path C) the shortest?
>
> Rose: Because…um…it's like, I could pull it down like this (indicating straightening out the path to form a single vertical segment), it would still be shorter than this because it would be kind of curvy like this (pointed to Path B).

Kyle and Rose's explanations here are consistent with the compression intuition because they talked about how the paths would compare if they straightened them out. In the following section I illustrate how multiple participants used this intuition, as well as the other three main intuitions, throughout the study.

Interactions Among Intuitions for Comparing Rectilinear Paths

Individual students exhibited interactions in their use of these four main types of intuitions in two different ways, combinations and conflicts. Some students applied intuitions in combination when defending their claims about their ordering of a particular set of rectilinear paths. In those situations, students used multiple intuitions to defend a single claim. Some students experienced conflicts in their intuition use. That is, they may have defended their claim about the ordering of a set of rectilinear paths through the use of one intuition, and then subsequently used a different intuition to justify a claim that the ordering of the paths should be different. The following sections illustrate how the 16 interview participants used intuitions in conflict or combination to support their claims about the order of the rectilinear paths (Tasks 1 and 2) by length.

Complexity and straightness in combination. Students used a combination of the complexity and straightness intuitions in combination a total of eight times across Tasks 1 and 2. For example, Ned (Grade 6, CLM Group) used the complexity and straightness intuitions in combination during Task 2 to defend why he thought Path C was the shortest saying, “Because it only has one turn and it almost goes straight to school.” Ned’s comment that Path C “only has one turn” is indicates that he was attending to the number of turns, or complexity, of the path. Because he followed this comment with “and it almost goes straight to school” indicates that he was also attending to the directness, or straightness, of Path C. Therefore, Ned’s response here indicates that he used both complexity and straightness intuitions to defend his placement of Path C as the shortest in the set.

Complexity and detour in combination. There were six instances of the use of the complexity and detour intuitions used in combination for Tasks 1 and 2. For example, Noah (Grade 4, CLM Group) used the complexity and detour intuitions to justify why he thought String 1 was the longest. He initially explained that String 1 is the longest because “this one goes down (tracing his finger down the vertical segment of String 1) and then that way (tracing his finger along the horizontal segment of String 1).” That is, Noah initially used the detour intuition. However, when asked another probing question, he relied on a combination of intuitions:

> Interviewer: Alright. What is it about going down and then that way (tracing finger along String 1) that makes String 1 the longest?
>
> Noah: cuz um...you would have a right angle here, and if you took a ruler, this one would be long (spanning fingers across the vertical segment of String 1) and that one would be long (spanning fingers along the horizontal segment of String 1)...um...and it doesn't have like a bunch of right angles like this one (pointing to String 3) does.

Noah’s initial attention to the lengths of the vertical and horizontal segments as making String 1 long suggests that he used the detour intuition. His follow-up statement about the “bunch of right angles” of String 3 is evidence that he also used the complexity intuition to justify his claim that String 1 was the longest.

Complexity and compression in combination. I observed one instance of the complexity and compression intuitions being used in combination across Tasks 1 and 2. For example, Kyle (Grade 10, ICPM Group) used the complexity and compression intuitions as a combination to defend his claim that Path D was the longest for Task 2:

> Interviewer: OK. And why is Path D the longest?
>
> Kyle: All of the separate lines adding them together (pointing to Path D), especially the last two lines that are much longer than they should be…but...uh...putting all of these lines (off camera pointing to Path D) and

straightening them out into one direction, it would just go farther than how I think Path A would go.

Kyle's initial attention to "all of the separate lines" is an indication that he initially was relying on the number of a specific feature of the paths, the number of line segments. This is evidence that he was initially using the complexity intuition. His next statements about "adding lines together" and "straightening them out into one direction" suggests that he was also thinking about straightening the paths, which is consistent with the compression intuition. Therefore, he used the complexity and compression intuitions in combination to defend a single claim about Path D being the longest in the set of rectilinear paths for Task 2.

Detour and straightness in combination. Nine instances of the use of the combination of the detour and straightness intuitions were observed in students' responses to Tasks 1 and 2. For example, Scott (Grade 10, ALM Group) used the detour and straightness intuitions as a combination to defend his claim that Path D was the longest for Task 2:

Interviewer: OK. And why is Path D the longest?

Scott: I think the length of D's short turns could be just one straight line or diagonal, and...hmmm...because if you make a right triangle with those it would be (traces finger as a diagonal from the beginning of one horizontal segment to the end of a vertical segment on path D)...the same as the...hmmm...

Interviewer: So are you imagining making a right triangle...will you show me what you're imagining with the triangle?

Scott: Like a right triangle the hypotenuse would be a shorter way...to get from A to B (traces finger as a hypotenuse from the beginning of the first horizontal segment of Path D at home to the end of the first vertical segment of path D) than...(traced finger along the first horizontal segment of path D and the first vertical segment of path D).

Scott's initial response was to make a diagonal line with all of the short turns in Path D. He then quickly switched from talking about making a diagonal to making a right angle with all of the segments of Path D. When asked to show what he was imagining with the triangle, Scott talked about the hypotenuse of the triangle being a short way to get from one point to another. This is consistent with the straightness intuition. His comparison of the short, straight hypotenuse way of getting from one point to another as being shorter than going between the same two points in an L-shaped path suggests that he was also using the detour intuition to defend his claim. Interestingly, Scott's mention of the hypotenuse of a triangle as being a shorter length than the sum of the legs of the triangle suggests that Scott's intuitive thinking for path length is integrated with his mathematical reasoning about right triangles.

Three intuitions in combination. One student used three intuitions to defend a claim. Rick (Grade 8, ICPM Group) used the complexity, detour, and straightness intuitions to explain why String 3 was shorter than String 1:

Interviewer: OK. Why is this one (pointing to String 3) shorter than this one (pointing to String 1)?

Rick: Um...because it (pointing to String 3) doesn't go like one long way all the way (traces an L-shape path on the String 3 transparency in the same shape as

String 1) it just goes like (traces along the segments of String 3) all the way...it goes like...it's like this (pointing to String 2). It's diagonal, but it's just a little bit longer because it goes out and down and out and down.

Rick's statement that String 3 doesn't go "like one long way all the way" while tracing the L-shape of String 2 with his finger indicates that he initially used the detour intuition. His follow-up comment about String 3 going "…like this (pointing to String 2). It's diagonal," suggests that he relied on the straightness intuition to support his argument that String 3 is straighter than String 1, so String 3 must be shorter than String 1. His final statement about String 3 being "just a little bit longer [than String 2] because it goes out and down and out and down" suggests that he used the complexity intuition to justify his claim that String 3 is only approximately straight, which makes it shorter than String 1, but not as short as String 2.

Compression and detour in combination. One student, Marie (Grade 10, ALM Group) used the compression and detour intuitions as a combination to defend a claim. Her responses also indicate that she experienced conflicts in the claims she made based on intuitions. The section below describes Marie's use of intuitions in combination and conflict.

Conflicting intuitions. Marie (Grade 10, ALM Group) exhibited conflicting intuitions when resolving Task 1. She had initially ordered the strings from shortest to longest as String 2, 1, and 3 when she re-examined her ordering using intuitive thinking:

Marie: I'm just stuck between these two on which one's the longest (pointed to Strings 1 and 3).

Interviewer: OK. Well, tell me what you're thinking about.

Marie: Because this one like has more like stopping and starting points (pointed to String 3). It doesn't go like as direct as like this one is obviously the most direct (pointed to String 2). But…um…and then this one only has like one other stopping point (traced around String 1). But I'm like trying to like imagine them bent out, and I'm not sure. This one might actually be the longest (pointed to String 1).

Marie first used the complexity and straightness intuitions as a combination. She first mentioned the complexity intuition to make a statement about the "stopping and starting points" for String 3. She then used the straightness intuition when describing String 2 as "obviously the most direct." Next, Marie exhibited the compression intuition when she talked about "trying to like imagine them bent out." This compression intuition seemed to inform her conclusion that String 1 must be the longest and overruled her initial conclusion that String 3 was the longest based on the complexity intuition.

Although Marie rejected her conclusion based on the complexity intuition, when she evoked the compression intuition on Task 1, she applied the rejected complexity intuition again in Task 2. She initially ordered the paths from left to right (from shortest to longest) as Paths C, A, D, and B:

Marie: I think that's pretty much it.

Interviewer: That's pretty much it?

Marie: um...these two might switch (pointed to Paths A and D). I'm just not sure.

Interviewer: Which two?

Marie: These two middle ones (pointed to Paths A and D).

Interviewer: OK. Tell me what you're thinking about those two middle ones.

Marie: Um...that this one (pointed to Path A) has like less like starting and stopping points like going around (pointed to Path D), but it (pointed to path A) also has like longer stretches so when lengthened out, it might end up actually being longer than this (pointed to Path D).

Marie used the complexity intuition when she compared Paths A and D by the number of “starting and stopping points.” Her next statement about the longer stretches of Path A are consistent with the detour intuition, which was in conflict with the claim she had just defended using the complexity intuition. Therefore, the complexity and detour intuitions were in conflict in this statement. She then evoked the compression intuition by talking about lengthening the paths out, in order to resolve this conflict between the complexity and detour intuitions. By doing so, Marie was able to reason about the size and number of segments in a path. She then changed her ordering of the paths as Path C, D, A, and B. Although the detour intuition alone (or even in tandem with the complexity intuition) was not convincing enough for Marie to make a decision about the order of Paths A and D, she used it again to defend why Path B was the longest.

Re-using rejected intuitions. Some students who experienced conflicting intuitions, and rejected one intuition in favor of another, later re-used a rejected intuition to defend a subsequent claim. Mia (Grade 4, CLM Group) did this on four separate instances. For example, when asked to compare Strings 1, 2, and 3 by their lengths she said:

Mia: um...well…these two are probably about the same length (pointed to Strings 1 and 3) because you could just make these straight (pointed to the first two segments of String 3) and then they would probably be about the same as this (pointed to String 1)...and if you made this one (pointed to String 1) bumpy it would probably be the same as this (pointed to String 3).

Mia's discussion of making String 1 straight or String 3 bumpy for the purpose of comparing them indicates that she used the compression intuition to defend her claim that the two strings were the same length. However, when asked to put the strings in order, she ordered them as String 2, 1, and 3. When asked about her order, she experienced a conflict in her intuitions:

Interviewer: Is this one the longest (pointing to String 3)?

Mia: I think...no.

Interviewer: No? Because why?

Mia: Because...(She switched the order of Strings 1 and 3 to reflect an ordering of String 2, String 3, and String 1 from left to right)...it is longer to get from here to here (traced finger along the two segments of String 1) than it is to get from here to here (traced finger along segments of String 3).

Interviewer: Oh, I see. So, a minute ago you said that these are the same (pointed to Strings 1 and 3). Are they the same?

Mia: No.

Interviewer: Or are they different?

Mia: They are different. This one is longer (pointed to String 1).

Mia initially engaged with the task by operating on the compression intuition to defend her claim that Strings 1 and 3 were the same by length. However, she later evoked the detour intuition when tracing and explaining, "it is longer to get from here to here (traced finger along the two segments of String 1) than it is to get from here to here (traces finger along segments of String 3." These intuitions conflicted and she rejected the conclusion that she had reached by using the compression intuition. That is, the detour intuition was predominant in her thinking here.

Although she rejected the compression intuition in favor of the detour intuition in this instance, she continued using both of those same intuitions when responding to further questions about her ordering of the strings:

> Interviewer: OK. What is it about String 2 that makes you think it's the shortest?
>
> Mia: cuz it's…um…just straight (pointed to String 2) and this one's bumpy (pointed to String 3) so then if this one (pointed to String 3) was straight then it would be a lot longer than this one (pointed to String 2).

Here, Mia defended her claim that String 2 is the shortest by using a combination of intuitions, straightness and compression. She initially operated on the straightness intuition when claiming that String 2 is just straight. She then elaborated and offered further justification by operating on the compression intuition, which she had previously rejected, when explaining that if String 3 were to be made straight, it would be "a lot longer" than String 2. She continued using the straightness, compression, and detour intuitions to address questions about her order of the paths:

> Interviewer: OK. um...why is String 1 the longest?
>
> Mia: cuz...um...it takes a lot longer to go down to a corner (traced along the vertical segment of String 1) and then over there (traced along the horizontal segment of String 1) than it does to just go straight through the middle (traced a finger over String 2).
>
> Interviewer: What is it about String 1 that makes you think it's the longest?

Mia: cuz...um...if you put both of these strings (pointing to both the vertical and horizontal segments of String 1) in a straight line, then it would be pretty long.

Interviewer: OK.

Mia: probably a lot longer than this one (points to string three) and obviously this one (pointed to String 2) 'cuz it's already straight.

Mia initially defended her claim that String 1 was the longest using a combination of intuitions, detour and straightness. She used the detour intuition first when she talked about how much longer it would take to go down to a corner (along the vertical segment) and then over (along the horizontal segment). She then evoked the straightness intuition by saying it takes longer and tracing String 1 than it does to just go straight along string two. When asked a clarifying question about what it is about String 1 that made her think it was the longest, she used the compression intuition. Although she had previously used the detour intuition to reject her initial determination (that Strings 1 and 3 were the same length) derived from the compression intuition, she used the detour and compression intuitions, with the straightness intuition to support her claim that String 1 is the longest.

For Task 2, Mia used all four intuitions, including intuitions in combination in some instances, as she defended her ordering of the Paths from shortest to longest by their lengths as Path C, A, D, and B. She also experienced conflicting intuitions and used rejected intuitions when defending her claim that Path B was the longest:

Interviewer: OK. Let's see. Why is Path B the longest?

Mia: Um...cuz it has to go...um...really far down and then it has to go far over there.

Interviewer: OK. Why does going far down and far over there make Path B the longest?

Mia: mmmm...hmmm...hmmm....

Interviewer: Can you explain it? Can you explain why going down far and going over far makes a path longer...makes a path long?

Mia: hmmm...actually I think this one (pointed to Path D) is longer than that one (pointed to Path B and switched Paths B and D, so the ordering was Path C, A, B, D)

Interviewer: Oh. OK. Can you tell me why you switched your order?

Mia: Because I think that if you put this one (traced along the horizontal segment of Path B) there (rotated end-to-end with the vertical segment of Path B, forming a single straight line), it would be about like this long or this big (indicated on the table where this single straight line would end up) as this one would be, if you stretched it out like that (pointed to Path D).

Interviewer: OK. Alright. So, was there something about one of these paths looked that made you switch the order?

Mia: um…this one (pointed to Path D) was all bumpy which meant it...took up more string, and this one was less bumpy and just straight (pointed to Path B).

Mia initially defended her claim that Path B was the longest by using the detour intuition saying that Path B is longest because it goes “really far down and then it has to go far over there.” When probed about why this feature of Path B makes it the longest, she exhibited conflicting intuitions. That is, she switched her ordering of the paths from CADB to CABD and explained that she switched because she imagined stretching the paths out, which indicates that she operated on the compression intuition. In this situation, the compression intuition was the predominant intuition over the detour intuition in Mia's thinking, despite the fact that she had previously used the detour intuition to reject a conclusion derived from the compression intuition in Task 1. When further pressed what it was about the paths that made her switch the order, she talked about how Path D was bumpy and therefore took up more string than Path B; this explanation was based on the complexity intuition.

Analytical Strategy Use for Rectilinear Paths

Twenty-three instances of analytical strategy use were exhibited by seven of the 16 interview participants when they compared sets of rectilinear paths. Six of these seven students who used analytical strategies also used the four intuitions described in the sections above to defend their orderings of the rectilinear "strings" or "paths" by their lengths. Only one student, Lynn (Grade 8, CRM Group) relied solely on analytical strategies to justify her claims when comparing the rectilinear "strings" or "paths" by their lengths. The 23 instances of analytical strategy use consisted of three types of physical comparison strategies and one strategy that involved projecting or translating segments of paths either vertically or horizontally. In the sections below I illustrate each of these four analytical strategies and describe how participants used them.

Indirect comparison using finger span. Students who used this indirect comparison strategy placed a finger span across a segment of one path and then placed the same finger span across a segment of another path. For example, Scott used the "indirect comparison using finger span" strategy on Task 2 after using the fist straightness intuition to defend his claim that Path C was the shortest, and then experiencing a conflict between the complexity and compression intuitions to justify his ordering of Path B as longer than Path A. When asked specifically about what he was thinking about Paths A and B, he said:

> Hmmm…this one's probably about the same length (spanned fingers across the vertical segments closest to school on both Paths A and C) so I would think this would be longer (traced finger along the first four segments of Path A) than that (pointed to the diagonal segment of Path C).

Scott's strategy of spanning his fingers across the vertical segments that were closest to the point labeled as "school" on both Paths A and C does not meet the definition of being an intuition. It is not an immediate, direct, and global solution. It is a solution derived from comparing parts of the paths indirectly, using a finger span.

The indirect comparison strategy using a finger span was observed a total of two times by two students, Rose (Grade 6, CRM Group) and Scott (Grade 10, ALM Group). Both Rose and Scott used this analytical strategy along with intuitions. Scott went on to use additional comparison and projection strategies to justify his claims about the order of the paths for Task 2.

Superimposed pairs of rectilinear paths to compare directly. Students who used this analytical strategy placed one transparency containing a "string" or "path" directly on top of another transparency containing a different "string" or "path." For example, Zane (Grade 10, ALM Group) used the analytical strategy of superimposing pairs of paths to compare the paths directly after being initially asked to compare the paths in Task 2 by their lengths. He superimposed Path A onto Path C. He then switched the positions of Path A and C, so the paths were then ordered as CBAD. Next, he switched the position of Paths B and A, so the paths were ordered as CABD. He again superimposed Path B onto Path D and said, “Got it.”

This direct comparison strategy involving superimposing pairs of rectilinear “strings” or “paths” was observed a total of nine times by six participants: Trent (Grade 6, CRM Group), Lynn (Grade 8, CRM Group), Rick (Grade 8, ICPM Group), Ruth (Grade 8, ICPM Group), Zane (Grade 10, ALM Group), Scott (Grade 10, ALM Group). Aside from Lynn, who used analytical strategies only throughout Tasks 1 and 2, the other five of these participants used this superimposition strategy along with intuitions to justify their claims about the ordering of the “strings” or “paths” by their lengths. Of these five, Rick was the only student who did not also use at least one other analytical strategy on Tasks 1 and 2.

Segment matching comparison strategy. Students who used the “segment matching comparison strategy” purposefully matched the segments of one “path” or “string” to the segments of another “path” or “string” when superimposing pairs of strings or paths to directly compare. For example, when initially asked to compare Strings 1, 2, and 3 by their lengths for Task 1, Trent (Grade 6, CRM Group) superimposed the transparency of String 3 onto the transparency of String 2 and then took them apart. He then placed each segment of String 1 over String 2 to compare directly and then took them apart again. He said, “OK. I found it.” When asked what he found, he explained that he thought string three was the longest and string two was the shortest.

Participants used the segment matching comparison strategy a total of six times by three students: Trent (Grade 6, CRM Group), Lynn (Grade 8, CRM Group), and Ruth

(Grade 8, ICPM Group). Both Trent and Ruth also used this analytical strategy along with intuitions, and other analytical strategies for Ruth.

Project to form right angle. Students who used the "project to form right angle" analytical strategy indicated that they compared rectilinear paths by imagining translating vertical segments horizontally (left and right) or horizontal segments vertically (up and down) to form a single right angle. For example, after using a combination of intuitions (straightness and detour) to justify why String 2 was the shortest, Ruth (Grade 8, ICPM Group) used the "project to form right angle" strategy when asked why she thought String 1 was the longest:

> Ruth: Cuz, for the middle one I kind of visually do it, whereas if I take this line, this line, this line (pointed to horizontal segments of String 3) and make it like a straight line over here (traced finger across the transparency for String 3 to indicate how long the three horizontal segments of String 3 would be if they were one segment.)
>
> Interviewer: mm-hmmm
>
> Ruth: It would be the same as...like this one (traced finger along the horizontal segment of String 1)…and then if it's like these three (traced finger along three vertical segments of String 3).
>
> Interviewer: OK
>
> Ruth: Oh! OK...so...OK OK OK.
>
> Interviewer: You can draw. You can write more on here if you want to. Do you want to write what you were imagining on there?

Ruth: So this one here (touched the marker at the end of the second horizontal segment of String 3 and then projected it end-to-end with the first horizontal segment) and this one (touched the marker at the end of the third horizontal segment of String 3 and then projected it end-to-end with the projected second horizontal segment of String 3)…and I was thinking these three (swept marker over each of the three vertical segments of String 3)...I'm confused now because it looks like they're the exact same length (looked at Strings 1 and 3).

Interviewer: OK. Tell me what you were doing with these three (pointed to each of the three vertical segments of String 3).

Ruth: And I would move these (touched the first vertical segment of String 3 with the marker) back over here, (placed marker at the end of the horizontal line representing the three projected horizontal segments and drew the first vertical segment perpendicular to this segment) so then this would go down here. This would go down here (touched the marker to the second vertical segment and then projected it end-to-end with the projected first vertical segment of String 3; this new segment touched the third vertical segment of String 3) and that would the exact same length.

After Ruth applied the project to form right angle strategy here to justify her claim that String 1 was the longest, she changed the ordering of the paths that she had initially defended using a combination of the straightness and detour intuitions. That is, her application of this analytical strategy created a conflict between the combination of intuitions she had used and this analytical strategy. This conflict led her to reject a claim she had initially defended using the combination of intuitions. Three different participants used the project to form right angle analytical strategy a total of six times: Ruth (Grade 8, ICPM Group), Zane (Grade 10, ALM Group), and Scott (Grade 10, ALM Group). Each student who used this analytical strategy also used other intuitions and analytical strategies to defend their claims about the order of a set of rectilinear paths.

Relating Intuition for Rectilinear Paths to the Levels of the Length LT

I tracked patterns of use of the four main intuitions, combinations of intuitions, analytical strategies, rejecting intuitions, and using rejected intuitions when comparing sets of rectilinear paths (Tasks 1 and 2) within and across the four length LT level groups. Figure 45 illustrates the frequency of the appearance of intuition and analytical strategy codes relevant to Tasks 1 and 2.

	Tasks 1 & 2									
	Straightness Intuition	Complexity Intuition	Detour Intuition	Compression Intuition	Total Number of Intuitions Used	Combination of Intuitions	Conflicting Intuitions	Rejected an Intuition	Used a Rejected Intuition	Analytical Strategy Use
CLM Group	12	19	10	6	47	6	2	2	4	0
CRM Group	7	15	6	2	30	6	0	0	0	10
ICPM Group	12	10	7	0	29	7	0	0	0	7
ALM Group	11	8	8	10	37	8	4	2	2	6
Totals	42	52	31	18	143	27	6	4	6	23

Figure 45. Patterns of intuition and analytical strategy use for comparing rectilinear paths within and across LT groups.

In Figure 45, within each column, the darkest shade indicates the LT group with the highest frequency of an intuition or analytical strategy code. The lightest shade indicates the LT group with the lowest frequency of an intuition or analytical strategy code. Figure 45 illustrates developmental patterns within and across LT groups for the use of the four main types of intuitions and overall analytical strategy and intuition use. In the sections below I describe these patterns, beginning with Table 5.

Table 5

Distribution of Each Intuition for Comparing Rectilinear Paths across Length LT Level Groups (Tasks 1 and 2)

	CLM Group	CRM Group	ICPM Group	ALM Group	Totals
Straightness	28.57%* (12)	16.67% (7)	28.57% (12)	26.19% (11)	42
Complexity	36.54% (19)	28.85% (15)	19.23% (10)	15.38% (8)	52
Detour	32.36% (10)	19.35% (6)	22.58% (7)	25.81% (8)	31
Compression	33.33% (6)	11.11% (2)	0.00% (0)	55.56% (10)	18
Totals	32.98% (47)	20.98% (30)	20.28% (29)	25.87% (37)	143

* 28.57% of the instances in which the straightness intuition was observed occurred with the participants who were classified as members of the CLM group.

Patterns of Intuition Use within LT Groups for Rectilinear Paths

I observed a total of 143 instances of intuition use in the 16 main participants responses to Tasks 1 and 2, which both involved comparing sets of rectilinear paths. For these two tasks, participants evoked the complexity intuition more often overall than any of the other three main types of path length intuitions observed during the study. This was followed by straightness, then detour, and finally the compression intuition. I observed this same overall trend for the frequency of the appearance of each of the four main intuitions for the CLM and CRM level groups. However, the participants in the ICPM and ALM level group exhibited a different pattern of intuition use. In the ALM and ICPM level groups, participants used the straightness intuition most often. This was followed by complexity, detour, and compression for the ICPM group, and compression, complexity, and detour for the ALM level group.

Patterns of Intuition Use across LT Groups for Rectilinear Paths

The complexity intuition was used most often by students in the CLM level group, with 19 instances, and decreased across the length LT groups as the level of sophistication increased. The detour intuition was used most often by the CLM group, and the use of this intuition was approximately evenly distributed across the CRM, ICPM, and ALM level group, The straightness intuition was used most often by the CLM group, and the compression intuition was used most often by the ALM group.

Students in the CLM level group, the lowest level of the LT for length measurement that was included in the present study, exhibited the highest number of instances of intuition use, for a total of 47. This group was followed by the ALM level group, the highest level of the length LT included in the study, for a total of 37. The CRM and ICPM level groups both exhibited approximately the same number of instances of intuition use, with 30 and 29 respectively. This suggests that there exist developmental patterns in intuition use across the levels of the length LT. Specifically, the types of intuition, as well as the frequency of use of intuition, changes across the levels of sophistication for length measurement.

Intuitive and Analytical Thinking in Combination and Conflict

Participants used intuitive and analytical thinking in combination and conflict when ordering rectilinear paths by length (Tasks 1 and 2). Table 6 illustrates the frequency with which these events occurred throughout Tasks 1 and 2.

Table 6

Distribution of Intuitions Used in Combination, Conflict, and in Tandem with Analytical Thinking for Comparing Rectilinear Paths (Tasks 1 and 2) across Length LT Groups

	CLM Group	CRM Group	ICPM Group	ALM Group	Overall Totals
Conflicting Intuitions	33.33%* (2)	0.00% (0)	0.00% (0)	66.66% (4)	6
Combination of Intuitions	22.22% (6)	22.22% (6)	25.93% (7)	29.63% (8)	27
Rejected an Intuition	50.00% (2)	0.00% (0)	0.00% (0)	50.00% (2)	4
Rejected Intuition Use	66.66% (4)	0.00% (0)	0.00% (0)	33.33% (2)	6
With Analytical Strategies	0.00% (0)	42.86% (6)	14.29% (2)	42.86% (6)	14

* 33.33% of the instances in which a conflict in intuition use was observed occurred with the participants who were classified as members of the CLM level group.

Intuitive and Analytical Thinking in Combination and Conflict within LT Groups

Participants in the CLM level group showed evidence of using intuitions in combination, experiencing conflicts among intuitions that lead to the rejection of an intuition, and also later used rejected intuitions. CLM level participants showed no evidence of using analytical strategies with intuitions when comparing rectilinear paths by their lengths. CRM and ICPM level participants used intuitions in combination and with analytical strategies. Participant at the ALM level showed evidence of using intuitions in combination, with analytical strategies, experienced conflicts among intuitions that lead to the rejection of an intuition, and also used rejected intuitions.

Intuitive and Analytical Thinking in Combination and Conflict across LT Groups

Only students in the lowest and highest LT level groups, the CLM and ALM levels, exhibited evidence of experiencing conflicting intuitions on Tasks 1 and 2. These six instances of conflicting intuitions appeared in the responses of three students: Mia (Grade 4, CLM Group), Scott (Grade 10, ALM Group), and Marie (Grade 10, ALM Group). The instances of intuitions used in combination appeared almost evenly across the four length LT level groups. None of the students in the CLM level group, the lowest LT level group included in the study, used analytical strategies with intuitions to resolve Tasks 1 or 2. The students in this LT level group relied solely on intuition to justify their claims about their orderings of rectilinear paths by length. Analytical strategy use in tandem with intuitions appeared most often at the CRM and ALM levels (6 instances) and dropped off for the level between those levels, ICPM (2 instances). Instances of students rejecting intuitions and using rejected intuitions appeared only within the highest and lowest LT level groups included in the study, the CLM and ALM groups. Although the instances of rejecting a claim based on an intuition were evenly dispersed across these two level groups (two instances in each of the CLM and ALM groups), students in the CLM level group exhibited evidence of returning to use rejected intuitions more often than students in the ALM level groups.

Instances of intuition use occurred within each of the four length LT level groups for the tasks involving comparing rectilinear paths (Tasks 1 and 2). However, not all of the length LT level groups showed evidence of using analytical thinking when making such comparisons. Table 7 below illustrates the distribution of intuitive and analytical thinking across the four length LT level groups for Tasks 1 and 2.

Table 7

Distribution of Intuition and Analytical Strategy Use for Rectilinear Paths (Tasks 1 and 2) across Length LT Level Groups

	CLM Group	CRM Group	ICPM Group	ALM Group	Totals
Intuitions	32.87%* (47)	20.98% (30)	20.28% (29)	25.87% (37)	143
Analytical Strategies	0.00% (0)	43.48% (10)	30.43% (6)	26.09% (6)	23

* 32.87% of the instances in which an intuition was observed for Tasks 1 and 2 occurred with the participants who were classified as members of the CLM level group (for a total of 47 instances).

Patterns of Intuition and Analytical Strategy Use within LT Groups

Within each LT level group, instances of intuitive thinking were more frequent than analytical thinking when comparing sets of rectilinear paths by lengths without tools. At the CLM level, students relied only on intuitions to defend their orderings of rectilinear paths by their lengths. At each of the subsequent levels, the CRM, ICPM, and ALM levels, students used both intuitions and analytical strategies to defend their claims about the order of rectilinear paths by their lengths.

Patterns of Intuition and Analytical Strategy Use across LT Groups

Table 7 indicates that, at the lowest level of the length LT included, the CLM level, students used the highest percentage of intuitions, with 32.87%. The highest percentage of analytical strategy use occurred within the group representing the next level, the CRM level, with 43.48%. This level group also exhibited one of the smallest percentages of intuition use, with 20.98%. Analytical strategy use then decreased as the length LT levels increased, with 30.43% for the ICPM group and 26.09% for the ALM group. Intuition use remained approximately the same from the CRM level group to the

ICPM level group, with 20.28%; however, there was an increase in intuition use for the ALM level group, with 25.87%.

This suggests an interaction between intuitive and analytical thinking for comparing sets of rectilinear paths with the levels of the length LT. This interaction is illustrated in Figure 46, where the blue bar represents the percentage of units of data within each LT group that were coded as intuitive, and the red bar represents the percentage of units of data that were coded as analytical.

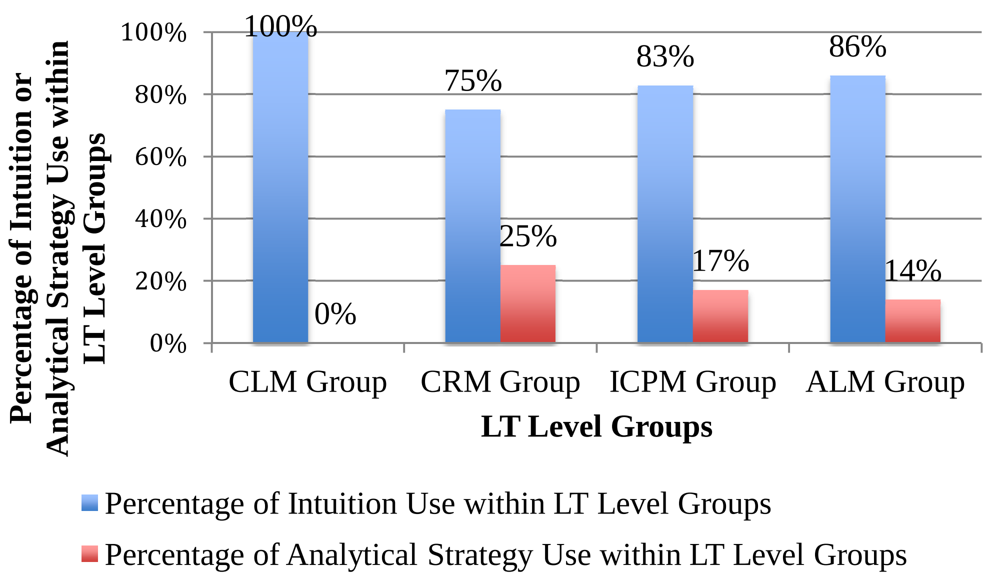


Figure 46. Interaction between intuitive and analytical strategy use within each LT level group.

Figure 46 indicates that, early on in the development of conceptual and procedural knowledge for length measurement, at the CLM level, students used only intuitive statements to defend claims about the order of rectilinear paths by length. Students at the subsequent level of conceptual and procedural knowledge for length measurement, the CRM level, used newly acquired analytical strategies to justify their claims about the lengths of rectilinear paths. Across the CRM, ICPM, and ALM levels, the ratio of intuition to analytical strategy use increases.

Curvilinear Paths: Intuitions and Analytical Strategies

I posed tasks involving curves to elicit students' intuitions and analytical strategies when comparing a pair or set of curves without tools (Task 6A, 7, and 8A), comparing a curve and a straight object (Tasks 3, 4, and 5), a pair of curves with a straight object (Tasks 6B and 8B), or measuring a curve with a ruler. In the first section below I describe the intuitions and analytical strategies that students used when comparing a pair or a set of curvilinear paths. This is followed by a section in which I describe relationships between intuition use and conceptual and procedural knowledge for length measurement, as measured by the LT for length measurement.

For Tasks 6A, 7, and 8A (Figures 47, 48, and 49) I asked students to compare a pair or set of curves without tools.

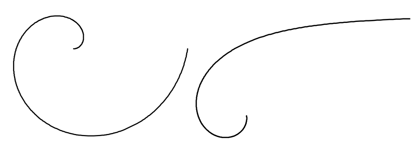


Figure 47. Image of a pair of curvilinear paths for interview Tasks 6A and 6B.

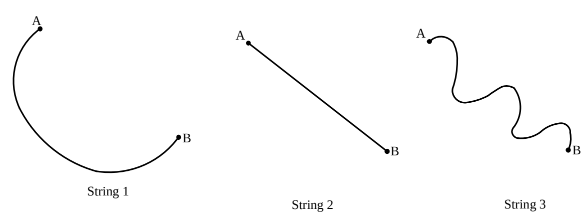


Figure 48. Image of a pair of curvilinear paths for interview Task 7.

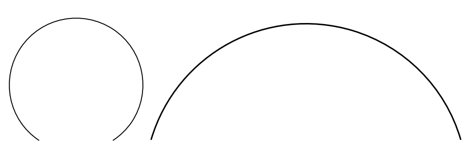


Figure 49. Image of a pair of curvilinear paths for interview Tasks 8A and 8B.

Five Intuitions for Curvilinear Paths

The 16 interview participants' responses to tasks involving ordering pairs or sets of curvilinear paths by their lengths without the use of tools (Tasks 6A, 7, and 8A) reflected five main intuitions. I observed four of these five intuitions in their responses as they compared rectilinear paths by their lengths (Tasks 1 and 2 in the sections above), as well as in prior research (Chiu, 1996). These are the straightness, complexity, detour, and compression intuitions. One new intuition, called the curve tightness intuition here, emerged as students attended to curve when telling about their ordering or answering clarifying questions about why a particular path was shortest or longest. In the following section I describe how the 16 interview participants used these five intuitions.

Straightness. As with the rectilinear paths, students who used the straightness intuition when defending their claims about the order of curvilinear paths by their lengths (Tasks 6A, 7, and 8A) attended to the straightness of a particular path, without providing further justification. For example, Kevin (Grade 4, CLM Group) used the straightness intuition when asked why the wider curve for Task 8A (see Figure 49) was the shortest. He said, "Because it's...it look...um...um...because it's more of a straight line than this one (pointed to the tighter curve)." Participants in each of the four length LT level groups reflected the straightness intuition in their responses across Tasks 6A, 7, and 8A.

Detour. Consistent with the use of this intuition for comparing sets of rectilinear paths (Tasks 1 and 2), students who used the detour intuition when defending their claims about the order of curvilinear paths (Tasks 6A, 7, and 8B) discussed a particular path as going out of the way or not being a direct route. For example, when asked to tell about his ordering of the three strings by their lengths for Task 7 (see Figure 48), Rick (Grade

8, ICPM Group) said, “I was thinking this is the longest (pointed to String 1) because you have to go all the way around.” I observed the detour intuition in students’ responses across all four length LT level groups; however, it only appeared in students’ responses to Task 7.

Complexity. Similar to some of the responses observed as students compared rectilinear paths (Tasks 1 and 2), students who used the complexity intuition to defend their claims about the ordering of curvilinear paths (Tasks 6A, 7, and 8A) attended to the number of turns of a particular path. For example, Grant (Grade 6, ICPM Group) used the complexity intuition when defending his claim that String 3 was the longest for Task 7:

> Interviewer: OK. And why is string three the longest?
>
> Grant: Because it like zig-zags all over and zig-zagging takes, like, extra time.
>
> Interviewer: OK. So, why does zig-zagging make it longer?
>
> Grant: Because you're like...you're like going all over the place instead of like straight from one point to another.

Grant’s response reflects his attention to the number of zig-zags in a path, as well as his belief that zig-zags add time and length to a path. His response illustrates the complexity intuition as applied to a set of curvilinear paths and a potential reason that people develop the complexity intuition, an interference of distance traveled versus the time it would take to travel that distance. For Task 7, Strings 1 and 3 are the same length. However, to traverse String 3 as a path in reality, one would need to slow down to make the turn, thus adding more time to the trip without adding length. Like the detour intuition, the complexity intuition was reflected in the responses of students in all four length LT level groups, but I observed it as they defended their claims about their orderings of the three curvilinear paths only for Task 7.

Compression. As with the sets of rectilinear paths (Tasks 1 and 2), students who used the compression intuition to justify their claims about the order of pairs or sets or curvilinear paths (Tasks 6A, 7, and 8A) discussed either straightening curves or bending curves for the purpose of making comparisons. For example, Scott (Grade 10, ALM Group) used the compression intuition to defend why he thought the wider curve was shorter for Task 8A saying, "just how I imagined if you...if it were a string you could just pull it (gestured to the wider curve with his fingers as if to pull the ends straight.)" The compression intuition was observed in students' responses for Tasks 6A, 7, and 8A and across all four length LT level groups.

Curve tightness. Students who used the curve tightness intuition discussed one curve as being longer than another because it was curved in more or had more curve. When asked how he thought about comparing the two curves for Task 6A, David (Grade 8, ICPM Group) said:

> I saw this one (pointed to the spiral curve) was more curved than this one (pointed to the curve with the straight segment). So, I was gonna see if...like...how close they are in length this way (gestured with his hands in a back and forth horizontal direction), and if this one (pointed to the spiral curve) was like the same size as this one (pointed to the curve with the straight segment), this one would be longer (pointed to the spiral curve) because it's curved in more. This one's slightly longer (pointed to the curve with the straight segment), but I still think that one's longer (pointed to the spiral curve).

David's response here is qualitatively different from the other four intuitions. He did not attend to the straightness of a path (the straightness intuition), or discuss straightening or bending one of the curves (the compression intuition). David also did not discuss one path as deviating away from the destination more than another (the detour intuition) or attend to the number of turns or bends in a path (the complexity intuition). Although his response here did not fit with the description of any of the four previously mentioned intuitions, his response is consistent with the properties of intuitions as described by Fischbein (1987). That is, it is a statement that is immediate, direct, and global. It was given without further justification or elaboration. This response from David is an illustrative example of the curve tightness intuition, which was repeatedly used in multiple curvilinear path length comparison situations (Tasks 6A, 7, and 8A) and by interview participants in each of the four length LT level groups throughout the study.

Analytical Strategy Use for Curvilinear Paths

Twenty-two instances of analytical strategy use were observed in the responses of nine of the interview participants as they made comparisons among curvilinear paths (Tasks 6A, 7, and 8A). Five of these nine students, Scott (Grade 10, ALM Group), Trent (Grade 6, CRM Group), Zane (Grade 10, ALM Group), Kyle (Grade 10, ALM Group), and Rose (Grade 6, ALM Group) each used only analytical strategies, without also using an intuition, to defend their order of the curvilinear paths for one of these tasks. Four students Lynn (Grade 8, CRM Group), Ruth (Grade 8, ICPM Group), Scott (Grade 10, ALM Group), and Kyle (Grade 10, ALM Group) used multiple analytical strategies across these tasks. One student, Lynn, used only analytical strategies on all of these tasks.

The following sections illustrate how participants used five different analytical strategies as they defended their orderings of sets of curvilinear paths for Tasks 6A, 7, and 8A.

Superimposed pairs of curvilinear paths to compare directly. Students who used this strategy placed one curve on top of the other for the purpose of directly comparing the strings. This strategy does not meet the definition of an intuition as an immediate, direct, and global approach because it involves a physical, direct comparison. Kyle (Grade 10, ALM Group) used this analytical strategy when comparing the tight curve and the wide curve for Task 7 (Figure 49). When initially asked to compare the set of three curvilinear "strings" by their lengths, Kyle placed the String 1 transparency on top of the String 2 transparency with points A and B lined up. Next, he placed the String 2 transparency on top of the String 3 transparency, again with points A and B lined up. He said, "String 2 is definitely the smallest because when I put it on top of 1 of…each of the other strings...(trailed off)." Although Kyle did not articulate how superimposing the strings informed his answer, the fact that he did suggests that he did not use an intuition.

A total of 14 instances of the analytical strategy of superimposing pairs of curvilinear paths to compare sets of curvilinear paths directly (Tasks 6A, 7, and 8A) were observed in seven participants' responses: Rose (Grade 6, CRM Group), Lynn (Grade 8, CRM Group), David (Grade 8, ICPM Group), Ruth (Grade 8, ICPM Group), Zane (Grade 10, ALM Group), Scott (Grade 10, ALM Group), and Kyle (Grade 10, ALM Group). Six of these seven participants who used this strategy, did so more than once. Once student, Ruth (Grade 8, ICPM Group) used this strategy on each of Tasks 6A, 7, and 8A.

Indirect comparison using finger span. Students exhibited the strategy of indirectly comparing using finger span by placing a finger span, or space pinched between fingers, on two or more curves. For example, Trent (Grade 6, CRM Group) used this strategy when he was asked to compare the partial circle-shaped curves for Task 8A without any tools. He placed one hand in an L-shape on each side of the tighter curve, with his thumbs touching. Next, he placed his hands in the same formation on top of the wider curve. He said, “I think this one is longer (pointed to the tight curve).” Kyle’s application of the indirect comparison using a finger span strategy was in his placement of a hand in a L-shape on each side of the tighter curve.

A total of three instances of the analytical strategy of indirect comparison using a finger span were observed in three student’s responses: Trent (Grade 6, CRM Group), Scott (Grade 10, ALM Group), and Kyle (Grade 10, ALM Group). These instances were observed on Tasks 6A and 8A.

Accumulating length comparison strategy. Students who used the accumulating length strategy superimposed a pair of curvilinear paths and rotated one of the paths, while accumulating the length of the first on a second path. Lynn (Grade 8, CRM Group) used this strategy when comparing the three strings by their lengths. She placed the String 1 transparency on top of the String 2 transparency, positioning String 2 as a tangent to String 1 and aligning them according to points A and B. Next, Lynn made a tick mark on String 1 at the point at which String 2 appeared to deviate from the curve. She then rotated String 1 on top of String 2, repositioning String 1 at a new point of tangency on String 2. Lynn again made a tick mark where String 2 appeared to deviate from the curve. This suggests that Lynn physically, by making tick marks, transformed one path into the same shape as another to compare them directly. Lynn repeated this procedure of adjusting the point of tangency, making tick marks to keep track, and accumulating the length of String 2 along String 1. She also applied this strategy for comparing Strings 1 and 3. She then ordered the strings from shortest to longest as String 2, 1, and 3.

I observed three instances of the accumulating length comparison strategy in three participants' responses: Lynn (Grade 8, CRM Group), Ruth (Grade 8, ICPM Group), and Scott (Grade 10, ALM Group). I observed all of the instances of this strategy on Task 7.

Rate comparison strategy. I observed one instance of the rate comparison strategy in one student's response to Task 7: Ruth (Grade 8, ICPM Group). She first superimposed the three "strings" to directly compare them. She then traced along the path of String 2 with a marker onto the String 1 transparency, and traced along String 1 on the String 2 transparency. Next, she placed String 3 onto String 1, aligned according to points A and B and traced the shape of Strings 1 and 2 on the String 3 transparency. She ordered the strings from shortest to longest as Strings 2, 3, and 1. While superimposing the strings, Ruth drew some marks (see Figure 50):

> Interviewer: I saw you making some marks on here (pointed to String 3) and then making some marks on there (pointed to the path of String 1 traced on the String 3 transparency).
>
> Ruth: I was...what I was doing is...I was kind of listening to it a little, and then look at it and...so that much right there (traced along a piece of String 3)...I tried to imitate that along...like right there (traced along a little piece of String 1 to show how part of String 3 mapped to String 1).

Here, Ruth's "listening" to "that much right there" (while tracing along a piece of String 3) and imitating "that much right there" (while tracing along a piece of String 1) suggests that she may have been attempting to traverse segments of the two paths while maintaining the same rate. That is, she compared the time it took to traverse a length on each string at the same rate as an attribute by which she could compare the curvilinear "strings."

Imposed internal unit. I observed one instance of the analytical strategy of imposing an internal unit when comparing curvilinear paths in one student's response for Task 7: Ruth (Grade 8, ICPM Group). While using the rate comparison strategy, Ruth made marks on the String 1, 2, and 3 transparencies, which suggests that she also applied an internal unit while comparing the curvilinear paths by their lengths for Task 7. The following figure (Figure 50) illustrates the tick marks the Ruth made while comparing the curvilinear paths by their lengths:

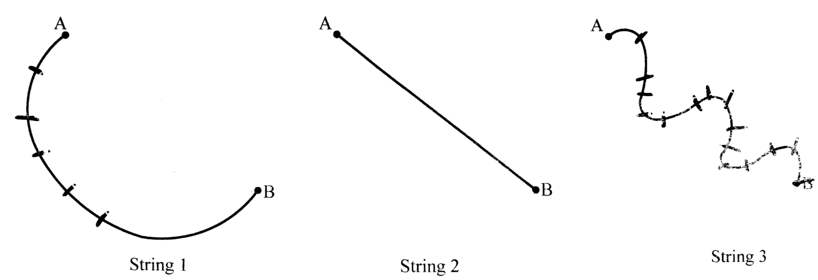


Figure 50. Ruth's application of an internal unit for Task 7.

> Ruth: So, what my thing would be was this amount right here would be equal to this amount right here (again, pointing to a segment of String 3 and showing how it mapped to a segment of String 1) and then I took a little bit from here and right there. So the little marks I made...like...to kind of chop it up a little bit. And, right here, this was 10 little tiny marks. And I tried to make sure my marks were the same as much as possible
>
> Interviewer: OK.

Ruth: I have doubt, but...I'm pretty sure that this is correct.

Interviewer: OK. So, this was 10 tiny marks within here (pointed to the segment of String 1 labeled as 10)?

Ruth: Yes.

Interviewer: OK. Got it. How did you know how big to make the marks? Cuz you made tiny marks on here (pointed to String 3)?

Ruth: Yeah.

Interviewer: OK.

Ruth: So, I did one right there, one right there...(showed how she made tiny marks on String 3)...they were roughly about this long...(then made marks along String 1).

Interviewer: That's where they are?

Ruth: yeah.

Interviewer: Got it.

Ruth: I don't know if that's exactly ten, but, yeah.

Ruth's partitioning of segments of Strings 1 and 3 each with tiny marks, which created same-size intervals on each "string," suggests that she applied an internal unit for the purpose of comparing the curvilinear paths.

Two of the analytical strategies that I observed as students compared curvilinear paths (Tasks 6A, 7, and 8A), I observed as students compared rectilinear paths by their lengths (Tasks 1 and 2): superimposed pairs of curvilinear paths to compare directly and indirect comparison using finger span.

Interactions Among Intuitions for Comparing Curvilinear Paths

I observed the same two types of interactions among intuitions for students' comparisons of rectilinear (Tasks 1 and 2) and curvilinear paths (Tasks 6A, 7, and 8A): intuitions in combination and conflict. While comparing curvilinear paths, students used intuitions in combination with other intuitions. However, all of the instances of intuitions in conflict that I observed as students compared the curvilinear paths, occurred as conflicts with analytical strategies or with conceptual and procedural knowledge for length measurement. The sections below I describe how the students used intuitions in combination with other intuitions, and how they experienced conflicts between intuitive and analytical thinking for path length.

Complexity and straightness in combination. Participants exhibited the complexity and straightness intuitions in combination five times in five different students' responses for Task 7 when comparing sets of curvilinear paths. For example, Kevin (Grade 4, CLM Group) ordered the three "strings" from shortest to longest as String 2, 1, and 3. He used the complexity and straightness intuitions in combination to explain why String 3 was the longest saying, "It's got to curve more so it goes out and then it's gotta keep on going out instead of going straight it's got like it goes out and that makes it a lot longer (tracing the shape of String 3 on the table)." Kevin's attention to the String 3 as one that has to "keep on going out" is consistent with the complexity intuition. He elaborated by saying, "instead of going straight," which is evidence that he also evoked the straightness intuition to defend the same claim; therefore, he used the complexity and straightness intuitions in combination.

Complexity and detour in combination. Three participants used the complexity and detour intuitions in combination three times as they justified their orderings of curvilinear paths. David (Grade 8, ICPM Group) ordered the "strings" for Task 7 as String 2, 3, and 1. When asked why String 1 is shorter than String 3 he said, "Because, even though it goes further away from them, it only takes one turn there and back (traced around String 1) and String 3 goes back and forth." David's attention to the String 1 going "further away from them" shows that he evoked the detour intuition. He elaborated by evoking the complexity intuition by talking about String 3 as going "back and forth." Therefore, David used two intuitions, complexity and detour, in combination.

Complexity and compression in combination. I observed one instance of the complexity and compression intuitions in combination when a student defended a claim about the order of the set of curvilinear paths for Task 7. Kyle (Grade 10, ALM Group) ordered the "strings" from shortest to longest as String 2, 1, and 3. He then used the complexity and compression intuitions in combination to defend his claim that String 3 was the longest:

> Interviewer: OK...And why is String 3 the longest?
>
> Kyle: Um...because it goes around and keeps on curving and curving until it gets to the point, and...um...with String 1, I...um...noticed that there would have been enough for...enough to cover A and B (traced from A to B on the String 1 transparency)...
>
> Interviewer: mm-hmm
>
> Kyle: ...and going straight line out from A and straight line out from B and then...uh...there would be a shorter amount...it would about go up to here (pointed

on the table to indicate where String 1 would end up if he were to straighten it and make it go through point B) with the excess part of the line, but with this (pointed to String 3) there would probably...I'm just kind of estimating that there would be...it would go out farther (pointed to the table to indicate where String 3 would end up if he were to straighten it and make it go past point B to show that it would go out further than String 1).

Kyle's initial attention to String 3 as one that "goes around and keeps on curving and curving" shows that he first evoked the complexity intuition. He then elaborated by discussing and indicating how far both Strings 1 and 3 would stretch if they were straightened out, which suggests that he also used the compression intuition. Kyle used both intuitions to justify the same claim; therefore, he used two intuitions in combination. Kyle's response here to Task 7 was the only instance of the use of the complexity and compression intuitions used in combination to defend the ordering of curvilinear paths.

Compression and straightness in combination. Four different participants used the compression and straightness intuitions in combination five times as they compared sets of curvilinear paths for Tasks 6A and 7. For example, Rick (Grade 8, ICPM Group) used the compression and straightness intuitions in combination to defend his claim that the spiral curve was longer than the curve with the straight segment for Task 6A (see Figure 47). When asked why he thought the spiral curve was longer he said, "Um...I don't know. It coils around more than this one (pointed to the string with the straight segment), which is just more straight (traced finger around the curve with the straight segment). This one seems (pointed to the spiral curve) longer because it...I don't know...just seems longer that way (traced finger around the spiral curve)." Rick's initial claim that the spiral curve "coils around more than" the curve with the straight segment is evidence that he initially evoked the compression intuition to defend his claim. He then elaborated by talking about the string with the straight segment as one that "is just more straight," which shows that he used the straightness intuition to defend the same claim. His use of the compression and straightness intuitions to defend the same claim that the spiral curve is longer for Task 6A suggests that he used the two intuitions in combination.

Compression and detour in combination. I observed one instance of the combination of the compression and detour intuitions in a student's justification for the ordering of the set of curvilinear paths for Task 7. Marie (Grade 10, ALM Group) ordered the "strings" from shortest to longest as String 2, 3, and 1. When asked to explain why String 1 is longer than String 3 she said, "I think cuz it goes so far around (traced around String 1), where this one (pointed to String 3) has little places where it goes around, but I think if we stretched them out, this one would still be shorter (pointed to String 3)." Marie's initial claim about String 1 as going "so far around" shows that she first used the detour intuition to explain why String 1 is long. Her next claim about stretching the strings out for the purpose of comparing shows that she also used the compression intuition to defend the same claim. She used the detour and compression intuitions to justify the same claim; therefore, she used them in combination.

Detour and straightness in combination. I observed two instances of the combination of the detour and straightness intuitions in two students' responses as they compared the set of curvilinear paths for Task 7. For example, when initially asked to compare the three "strings" by their lengths Ruth (Grade 8, ICPM Group) said, "I know String 2 is shortest because it's the straightest path to go there, and this one goes all the way around (traced finger around String 1)." Ruth's statement about String 2 being the shortest because it is the straightest shows that she first evoked the straightness intuition to defend her ordering. She then turned to String 1, which she said "goes all the way around." This suggests that she also operated on the detour intuition to defend the same claim. Therefore, Ruth used detour and straightness in combination.

Curve tightness and straightness in combination. One student's response reflected the combination of the curve tightness and straightness intuitions when defending the ordering of a set of curvilinear paths. Noah (Grade 4, CLM Group) used this combination of intuitions to justify his ordering of the set of three curvilinear "strings" by their lengths for Task 7. He provided an ordering of the "strings" from shortest to longest as String 3, 1 and 2. I then asked a series of follow-up questions to probe Noah's thinking with respect to why he thought String 3 was the shortest and String 2 was the longest:

> Noah: Because there's all these curves (trailed off).
>
> Interviewer: OK. What is it about all those curves that makes you think it's the shortest?
>
> Noah: Um...
>
> Interviewer: Or, what is it about...why do curves make a string short?
>
> Noah: Um...because...um...um...like the curves make it shorter because...um...if you were measuring it from just like a straight line like String 2, um...it would be the easiest to walk cuz it would just be one straight solid line.

Noah's attention to curves making a "string" shorter because walking a straight line would be easier to walk than a curve is evidence that he evoked the curve tightness intuition. Because his thinking about why a straight solid line segment that connects two points is shorter than a curved line segment that connects the same two points was unclear, I asked further clarifying questions:

> Interviewer: OK. So, can you tell me again why you think the curves make the path short or make the string short?
>
> Noah: Because...um...if you had to walk it you would have to make all of the curves...like walk them
>
> Interviewer: OK. And that makes the path...makes the string shorter?
>
> Noah: mm-hmm
>
> Interviewer: OK. Why is String 2 the longest?
>
> Noah: Because it's just one straight solid line and there's no curves, so you can just walk it straight.

Noah's explanation that String 2 is "just one straight solid line" indicates that he used the straightness intuition. He elaborated by saying "and there's no curve," which indicates that he also evoked the curve tightness intuition in combination with the straightness intuition to defend his claim that String 2 is the longest. Noah's response to Task 7 was the only instance of the combination of the curve tightness and straightness intuition.

Conflicting Intuitions

Only two instances of intuitions in conflict were observed as students compared the set of curvilinear paths for Task 7. For this task, one student, Noah (Grade 4, CLM Group) experienced a conflict between a combination of intuitions and an important unit concept for length measurement. Another student, Ruth (Grade 8, ICPM Group) experienced a conflict between an intuition and an analytical strategy. The sections below illustrate Noah and Ruth's intuitions for curvilinear paths in conflict.

Conflict between intuitions in combination and a unit concept. Noah (Grade 4, CLM Group) experienced a conflict between a combination of the straightness and curve tightness intuitions and a key unit concept for measurement. Noah initially evoked the curve tightness and straightness intuitions in combination to defend his ordering of the three "strings" for Task 7 as String 3, 1, and 2. None of the other participants claimed that String 2, the straight string, was the longest. His explanation that String 2 is the longest because it is "just one straight solid line and there's no curves, so you can just walk it straight" was unclear. Therefore, I broke the interview protocol to further probe his intuitive thinking about the lengths of this set of curvilinear "strings" by asking him to imagine comparing them by the length of wire, yarn, or number of steps it would take to span each of the paths:

> Interviewer: Alright. Which one of these strings, if they were paths, which one would take the most steps?
>
> Noah: Probably String 3 (pointed to String 3).
>
> Interviewer: OK. Which would take the fewest steps?
>
> Noah: Probably String 2 (pointed to String 2).
>
> Interviewer: OK. And if you really were to...have you ever heard the story about Hansel and Gretel?
>
> Noah: yeah
>
> Interviewer: What do you remember about the story?
>
> Noah: They left a trail...

Interviewer: Trail that's exactly what I was thinking about. Of what?

Noah: I forgot.

Interviewer: Candy or something. They left a trail. Wouldn't it be fun if it were candy? Maybe I'm just hungry for candy. I don't remember if it was candy. What if you were walking each of these paths from A to B and you were leaving a trail of like wire behind you or yarn, which one would take the most yarn?

Noah: String 3.

Interviewer: OK. Which one would take the least amount of yarn?

Noah: String 2.

Interviewer: OK. So, I'm going to ask, I promise this is the last time, which string is longest?

Noah: String 2.

Interviewer: OK. Which string is shortest?

Noah: String 3.

When Noah initially thought about the set of curvilinear paths as strings, he ordered them from shortest to longest as String 3, 1, and 2. However, when I asked him to think of them as if they were wire, yarn, or the number of steps that it would take to span each of the paths, he changed his order (from shortest to longest) as String 2, 1, and 3. After this, once again he claimed that String 2 was the longest and String 3 was the shortest. That is, Noah was willing to change his ordering from Strings 3, 1, 2 to Strings 2, 1, 3 as I changed the context from curved “strings” to compare to curved “paths” to traverse and compare by the number of steps walked or the amount of wire or yarn left behind.

When thinking about Strings 1, 2, and 3 as "paths" to traverse, Noah was willing to say that String 2 would be "easiest to walk," String 3 would take the most steps, and insistence that String 2 was the longest. This suggests that he may have attended to number of steps he thought would be needed to walk each path without thinking those steps needed to be the same size to give a valid comparison. Specifically, I interpreted his response to my questions to indicate that he may have thought that a person would have to take a large number of small steps to walk a path with many turns (String 3), but one could take a small number of large steps to walk a straight path (String 2). Although Noah's intuitive thinking about the "strings" using the straightness and curve tightness intuitions conflicted with his thinking about the same objects as spanned by wire or yarn, or even unitized by steps, both ideas seemed to coexist as part of Noah's intuitive and conceptual knowledge for length measurement.

Conflict between an intuition and an analytical strategy. Ruth (Grade 8, ICPM Group) experienced a conflict between an intuition and an analytical strategy for comparing the set of curvilinear paths in Task 7. She initially ordered the "strings" as String 2, 1, and 3. She then superimposed String 3 onto String 1, aligning them according to points A and B. Ruth then said:

> I know String 2 is shortest because it's the shortest path to go there and this one is all the way around (traced finger around String 1) and this one kinda is longer because (traced finger around string three) it goes straight there but then it takes like extra path, where this is just straight (traced finger again along String 2), so I know this is the shortest one. The thing is deciding between String 1 and String 2...wait String 1 and String 3. From first glance, it looks like they are the same

> because, like, if you visually do it, and you take this out (gestured with figures as though to re-shape String 3 to make it the same shape as String 1) to make it a circle like this one (traced along String 1) it looks like it would be the same. But then, at the same time, I think it could be a little bit different. I wanna go with...(superimposed String 3 onto String 1 and rotated String 3 along String 1, accumulating the length of String 1 onto String 3).

Ruth evoked the compression intuition when she showed how she would "visually do it" by gesturing with her fingers how she was imagining re-shaping String 3 to make it the same shape as String 1. Using the compression intuition, she concluded "at first glance" that Strings 1 and 3 were the same. She then applied an analytical strategy of superimposing String 3 onto String 1, aligning the two strings according to points A and B. After comparing the strings directly by rotating String 3 along String 1 and accumulating the length of String 1 onto String 3, Ruth experienced a conflict with her initial conclusion that Strings 1 and 3 were the same length, which was derived from the compression intuition. Based on her conclusion from applying the analytical strategy, she changed her order of the curved "strings" from shortest to longest as String 2, 3, and 1. That is, Ruth rejected her conclusion derived from the compression intuition in favor of her conclusion derived from an analytical strategy, superimposing the strings to directly compare them. Ruth used the rejected compression intuition again when comparing two curves for Task 8A.

Relating Intuitions for Curvilinear Paths to the Levels of the Length LT

I tracked the developmental patterns for the five main intuitions, combinations of intuitions, analytical strategies, and rejected intuitions for comparing curves (Tasks 6A, 7, and 8A) within and across LT level groups. Figure 51 shows the frequency of intuitions and analytical strategies used. In each column, the darkest shade indicates the highest frequency and the lightest shade shows the lowest frequency of an intuition.

	Tasks 6A, 7, and 8A										
	Straightness Intuition	Complexity Intuition	Detour Intuition	Compression Intuition	Curve Tightness Intuition	Total Intuition Use	Combination of Intuitions	Conflicting Intuitions	Rejected an Intuition	Used a Rejected Intuition	Analytical Strategy Use
CLM Group	10	4	3	7	3	27	5	1	0	0	0
CRM Group	5	3	1	11	1	21	3	0	0	0	6
ICPM Group	11	8	4	5	3	31	7	1	1	1	8
ALM Group	5	1	2	8	1	17	3	1	0	0	8
Totals	31	16	10	31	8	96	18	3	1	1	22

Figure 51. Patterns of intuition and analytical strategy use for comparing curvilinear paths within and across LT groups.

Figure 51 depicts the patterns for intuitive and analytical thinking for comparing curves within and across the groups. Next, I describe the nature of these patterns, beginning with intuitive thinking for tasks involving comparing curves without tools in Table 8 below.

Table 8

Distribution of Each Intuition for Comparing Curvilinear Paths (Tasks 6A, 7, and 8A) within and across Length LT Level Groups

	CLM Group	CRM Group	ICPM Group	ALM Group	Totals
Straightness	32.26%* (10)	16.13% (5)	35.48% (11)	16.13% (5)	31
Complexity	25.00% (4)	18.75% (3)	50.00% (8)	6.25% (1)	16
Detour	30.00% (3)	10.00% (1)	40.00% (4)	20.00% (2)	10
Compression	22.58% (7)	35.48% (11)	16.13% (5)	25.81% (8)	31
Curve Tightness	33.33% (3)	11.11% (1)	44.44% (4)	11.11% (1)	9
Totals	27.84% (27)	21.65% (21)	32.99% (32)	17.53% (17)	97

* 32.26% of the instances in which the straightness intuition was observed over Tasks 6A, 7, and 8A occurred with the participants in the CLM group.

Patterns of Intuition Use within LT Groups for Curvilinear Paths

I observed 97 instances of intuition use in the 16 interview participants' responses to curvilinear path comparison tasks (Tasks 6A, 7, and 8A). For these tasks, participants evoked the straightness and compression more often than any other intuition. This was followed by complexity, then detour, and finally the curve tightness intuition. For the CLM group, I observed the straightness intuition most often; this was followed by compression, complexity, and the same number of instances of detour and curve tightness. Participants in the CRM group used the compression intuition most often, followed by straightness, complexity, and one instance each of detour and curve tightness. In the ICPM group, the most frequently used intuition was straightness, which was followed by complexity, compression, the same number of occurrences of detour and curve tightness. The ALM group used the compression intuition most often, followed by straightness, detour, and one instance each of complexity and curve tightness.

Patterns of Intuition Use across LT Groups for Curvilinear Paths

The CRM and ALM groups exhibited almost the same pattern of intuition use as was observed with the entire sample. The CLM and ICPM groups exhibited patterns of intuition use that were similar to each other, but different from the entire sample. Both the CLM and ICPM groups exhibited the straightness intuition most often. This was followed by the complexity and compression intuitions (in reverse order for CLM) and then the same number of instances of the detour and curve tightness intuitions. The straightness, complexity, detour, compression, and curve tightness intuitions were used most often in the ICPM group. The compression intuition was used most often at the CRM level. The ICPM group exhibited the highest frequency of intuition use.

Intuitions for Curves in Combination, Conflict, and with Analytical thinking

As was the case when participants compared rectilinear paths by their lengths (Tasks 1 and 2), when comparing curvilinear paths by their lengths (Tasks 6A, 7, and 8A), participants sometimes used analytical strategies or intuitions in combination. Some responses also suggested a conflict between conclusions drawn from intuitions and those drawn from analytical strategies, whereas other responses indicated that intuitions were used with analytical strategies. Table 9 shows how intuitions were used in combination, conflict, and with analytical strategies for comparing curves.

Table 9

Intuitions in Combination, Conflict, and in Tandem with Analytical Thinking (Tasks 6A, 7, and 8A) across Length LT Level Groups

	CLM Group	CRM Group	ICPM Group	ALM Group	Overall Totals
Conflicting Intuitions	50.00% (1)	0.00% (0)	50.00% (1)	0.00% (0)	2
Combination of Intuitions	27.78% (5)	16.67% (3)	38.89% (7)	16.67% (3)	18
Rejected an Intuition	0.00% (0)	0.00% (0)	100.00% (1)	0.00% (0)	0
Rejected Intuition Use	0.00% (0)	0.00% (0)	100.00% (1)	0.00% (0)	0
With Analytical Strategies	0.00% (0)	10% (1)	50.00% (5)	40.00% (4)	10

* 50.00% of the instances in which a conflict in intuition use was observed when comparing curvilinear paths (Tasks 6A, 7, and 8A) occurred with CLM level participants.

Intuitive and Analytical Thinking in Combination and Conflict within LT Groups

Participants in the CLM group showed evidence of using intuitions in combination and experiencing a conflict among intuitions, but showed no evidence of rejecting intuitions, using a rejected intuition, or using an intuition with an analytical strategy. CRM and ALM level participants used intuitions in combination and with analytical strategies. At the ICPM level, participants showed evidence of using intuitions in combination, with analytical strategies, experienced conflicts among intuitions that lead to the rejection of an intuition, and used rejected intuitions.

Intuitive and Analytical Thinking in Combination and Conflict across LT Groups

One student each in the CLM and ICPM level groups exhibited conflicting intuitions when comparing curves for Tasks 6A, 7, and 8A: Noah (Grade 4, CLM Group) and Ruth (Grade 8, ICPM Group). The instances of combinations of intuitions appeared most often in the ICPM level group, which was immediately followed by the CLM group and the CRM and ALM groups. I observed one instance of rejecting a claim made based on an intuition, the compression intuition, at the ICPM level by Ruth (Grade 8). She was the only student who later went on to use this rejected intuition when comparing a different pair of curves. Intuitions used with analytical strategies appeared most often at the ICPM and ALM levels, with five and four instances, respectively. One instance of using intuitions and analytical strategies to compare curves appeared in the CRM group.

The use of intuitions when comparing curves (Tasks 6A, 7, and 8A) occurred within each length LT level group. However, only some of the LT groups evoked analytical thinking when comparing curves. Table 10 below illustrates the distribution of intuitive and analytical thinking across the LT groups for Tasks 6A, 7, and 8A.

Table 10

Intuition and Analytical Strategy use for Comparing Curvilinear Paths (Tasks 6A, 7, and 8A) across Length LT Level Groups

	CLM Group	CRM Group	ICPM Group	ALM Group	Total
Intuitions	27.84%* (27)	21.65% (21)	32.99% (32)	17.53% (17)	97
Analytical Strategies	0.00% (0)	27.27% (6)	36.36% (8)	36.36% (8)	22

* 27.84% of the instances in which an intuition was used when comparing curvilinear paths (Tasks 6A, 7, and 8A) occurred with the participants who were in the CLM group.

Patterns of Intuition and Analytical Strategy Use within LT Groups

In each LT level group, the intuitions were used more often analytical strategies when comparing curves without tools (Tasks 6A, 7, and 8A). At CLM level, participants only showed evidence of using intuitions to defend their orderings of curvilinear paths by length. Intuition use dropped slightly for the subsequent length LT level group. At the CRM, ICPM, and ALM levels, participants used both intuitions and analytical strategies.

Patterns of Intuition and Analytical Strategy Use across LT Groups

Table 10 shows that, the highest percentage of intuition strategy use occurred at the ICPM level, with 32.99%. Intuition use was at a minimum at the highest level, the ALM group, with 17.53%. Analytical strategy use increased from the CRM level group, at 27.27%, to the ICPM level group at 36.36%. This level of analytical strategy use was maintained at the highest level with the ALM level group. These results suggest that there exists an interaction between intuitive and analytical thinking for comparing sets of curvilinear paths with conceptual and procedural knowledge for length measurement, as measured by the length LT. At the lowest length LT level that was included in the study, the CLM level, students relied only on intuitions to justify orderings of curvilinear paths by their lengths. Students at the subsequent levels of the length LT, the CRM and ICPM levels, used analytical strategies along with intuitive statements, with approximately the same ratio of intuitions to analytical strategies, to defend their claims when comparing curvilinear paths by their lengths. By the highest level, the ALM level, the ratio of intuitions to analytical strategies decreased. The nature of this interaction is illustrated in Figure 52. In this figure, the blue bar indicates the percentage of the units of data occurring within each length LT level group, which I coded as intuitions, and the red bar indicates the percentage of those units of data, which I coded as analytical strategies.

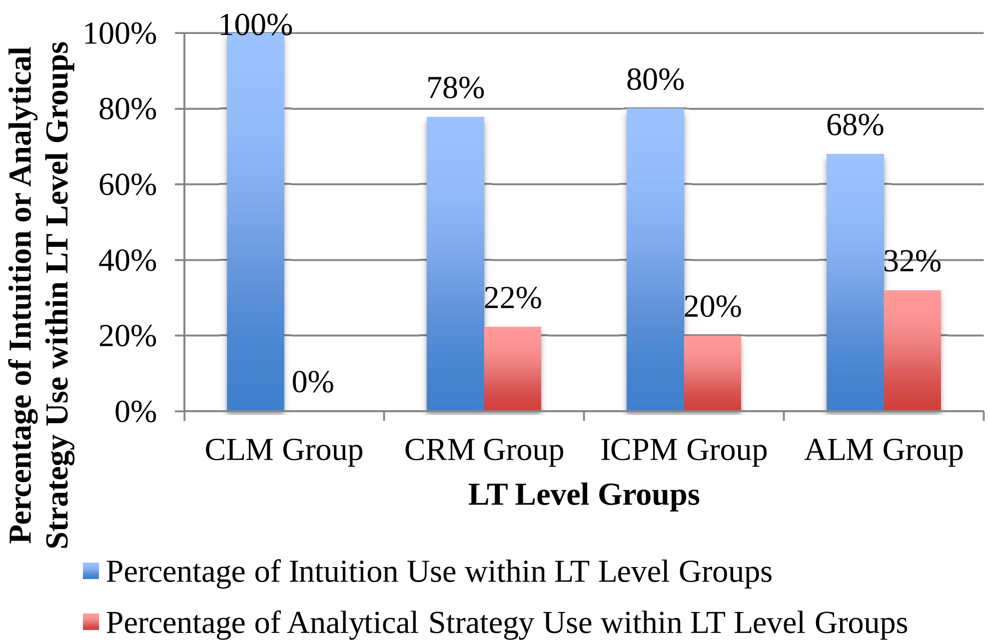


Figure 52. Interaction between intuitive and analytical strategy use for comparing curvilinear paths (Tasks 6A, 7, 8A) within each LT level group.

Comparing Curves and Straight Objects: Intuitions and Analytical Strategies

I posed five tasks (Tasks 3, 4, 5, 6B, and 8B) to probe students' intuitive and analytical thinking for curvilinear paths by asking them to compare a curve and a straight object. In the following sections I describe the intuitions and analytical strategies that students used to make these comparisons and reflect on the error in their comparisons of curves and straight objects. The first section describes the analytical strategies that students used to compare a curvilinear path to a straight object (Tasks 3, 4, and 5) or to indirectly compare two curvilinear paths using a straight object (Tasks 6B and 8B). Next, I illustrate how students used intuitive thinking when making such comparisons and reflecting on the error involved with their ways of comparing. Finally, I relate all of the intuitions and analytical strategies that students used when comparing curvilinear paths with straight objects to the levels of the LT for length measurement.

Analytical Strategies for Comparing Curvilinear Paths to Straight Objects

I observed analytical strategies as students compared a curve and a straight object (Tasks 3, 4, and 5) or indirectly compared two curves with a straight object (Tasks 6B and 8B). Figures 53, 54, and 55 illustrate the curves for Tasks 3, 4, and 5.

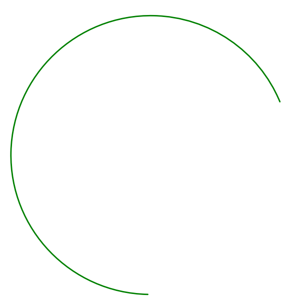


Figure 53. Image of curve shown for interview Task 3.

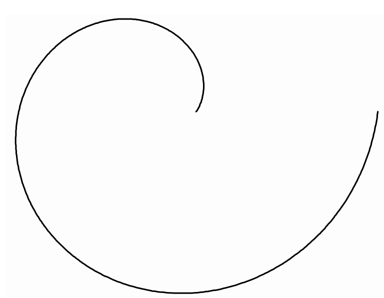


Figure 54. Image of curve shown for interview Task 4.

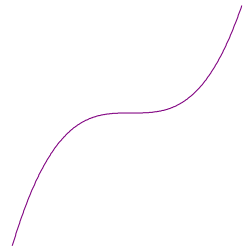


Figure 55. Image of curve shown for interview Task 5.

See Figures 47 and 49 for the images of the curves shown to students for Tasks 6B and 8B. In the following sections I describe how students used analytical strategies for comparing curves and straight objects (Tasks 3, 4, 5, 6B, and 8B).

Chord iteration strategy. There were 51 occurrences of the chord iteration strategy across Tasks 3, 4, 5, 6B, and 8B. Trent (Grade 6, CRM Group) used the chord iteration strategy when using a straight stick to compare two partial circle-shaped curves indirectly for Task 8B. For example, when measuring the tighter curve, he placed the stick as a chord aligned with one end of the curve and traced along the edge of the stick that was closest to the curve. Next, Trent repositioned the stick as a chord and aligned with this tick mark; he again traced along the edge of the stick closest to the curve. He repeated this procedure for a third full stick unit and one partial stick unit. He used the same strategy to compare the stick to the tighter curve; however, three stick units fit perfectly inside the curve (see Figure 56 below).

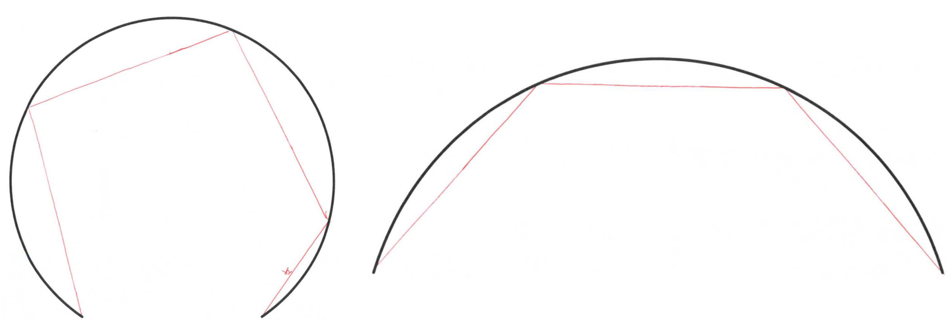


Figure 56. Trent’s chord iteration strategy for indirectly comparing two curves.

Continuous comparison strategy to estimate. The continuous comparison strategy was used by students to estimate a total of four times for Tasks 3, 4, 5, 6B, and 8B. For example, Mia (Grade 4, CLM Group) used the continuous comparison strategy to initially estimate the length of the spiral-shaped curve for Task 4. She placed the stick as a chord on the inside of the curve aligned with one end of the curve and then iterated the stick three times without using a finger or drawing a mark to keep track. Mia then moved the stick in a continuous motion for the final stick unit iteration. Mia's movement of the stick in a continuous motion suggests that she was using a continuous strategy for comparing by estimating rather than spanning the curve with stick units.

Tangent iteration strategy. Fourteen instances of the tangent iteration strategy appeared in students' responses to Tasks 3, 4, 5, 6B, and 8B. After initially using the continuous comparison strategy to estimate, Mia (Grade 4, CLM Group) used the tangent iteration strategy to compare the stick to the spiral-shaped curve for Task 4. She placed the stick as a tangent, aligned with one end of the curve. Next, Mia traced along the edge of the stick furthest away from the curve. She repositioned the stick, again placing it as a tangent along the outside of the curve and aligned with the segment representing her previous placement of the stick. She applied this strategy to the entire curve (Figure 57).

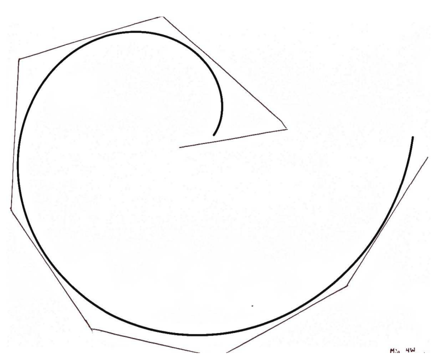


Figure 57. Mia's tangent iteration strategy for comparing a curve and a straight object.

Mixed unit iteration strategy. Eight instances of the mixed unit iteration strategy were observed in students' responses to Tasks 3, 4, 5, 6B, and 8B. Kevin (Grade 4, CLM Group) used a mixed unit iteration strategy when comparing a straight object to a partial circle-shaped curve for Task 3. He aligned the stick with one endpoint of the curve and iterated it around the curve, sometimes laying the stick as a chord and sometimes laying the stick directly on the curve. He said that the curve was three sticks longer than the stick.

Path intersection iteration strategy. The path intersection iteration strategy was observed in three responses for Tasks 3, 4, 5, 6B, and 8B. David (Grade 8, ICPM Group) used this strategy when comparing a stick to an S-shaped curve for Task 5. David placed the stick on the curve, aligned with one endpoint of the curve. He then repositioned the stick to the opposite end of the curve, again placing it on the curve aligned with one endpoint of the curve. He then drew three stick-sized rectangles on the curve to represent how he had either physically or mentally placed the stick on top of the curved path.

Adjusting point of tangency iteration strategy. There were 11 occurrences of the adjusting point of tangency iteration strategy across Tasks 3, 4, 5, 6B, and 8B. For example, Ruth (Grade 8, CRM Group) used the adjusting point of tangency iteration strategy to compare a stick to a partial circle-shaped path for Task 3. She placed the stick as a tangent aligned with one endpoint of the curve, and she rotated the stick while changing the point of tangency and accumulating the length of a segment of the curve along the stick. Ruth then drew a tick mark where the end of the stick touched the curve. Next, she repositioned the stick, aligning one end of the stick with this tick mark and again applied the strategy of rotating the stick while changing the point of tangency and accumulating the length of a segment of the curve along the stick. She repeated this procedure for a total of four full stick units and one partial stick unit. She wrote "about four and one third" on the page.

Modified circumference formula strategy. I observed five instances of applying an algorithmic approach, a modified circumference formula, in students' responses to Tasks 3, 4, 5, 6B, and 8B. Two students, David (Grade 8, ICPM Group) and Zane (Grade 10, ALM Group) used this strategy in their responses to Tasks 3, 4, and 8B. For example, on Task 3 Zane placed the stick on the interior of the curve positioned as a radius. He said the curve "is two thirds the circumference of the whole...if it was a whole circle." He then said the length of the curve would be "two thirds two pi R," where "R" is the length of the stick.

Analytical Strategies Related to Unit

I observed different analytical strategies related to the ways in which students operated on the nonstandard unit, the 4-inch stick, when comparing a curve and a straight object (Tasks 3, 4, and 5) or indirectly comparing two curves with a straight object (Tasks 6B and 8B).

Used the whole stick as a unit. Participants applied the strategy of using the whole stick as a unit when comparing a curve and a straight object (a stick) a total of 75 times over Tasks 3, 4, 5, 6B, and 8B. Marie (Grade 10, ALM Group) used this strategy when indirectly comparing two rectilinear paths with a straight object (a stick) for Task 6B. For the spiral-shaped curve, she placed the stick as a chord, aligned with one end of the curve. Marie traced along the edge of the stick closest to the curve, and then she repositioned the stick as a chord, placing it at the intersection of the line segment representing the previous chord stick unit and the curve. As she used the stick to compare the two curves indirectly, she applied this chord stick unit iteration strategy, using the whole stick as a unit, for a total of four stick units.

Fractured non-standard unit once at the endpoint of the curve. Students fractured the non-standard unit, the stick, when another full stick unit did not fit along the curve at the end, 31 times for Tasks 3, 4, 5, 6B, and 8B. For example, Marie (Grade 10, ALM Group) also applied this strategy when using the stick to measure the spiral-shaped curve during Task 3 (described in the previous paragraph). After applying the fourth full stick unit, part of the curve extended beyond the end of the stick. Rather than ignoring this remaining segment of curve, Marie quantified it by fracturing the stick unit, saying the curve "has about four and then...like...another like probably three fourths" stick units.

Fractured non-standard unit in the tightest part of the curve. I observed the strategy of increasing precision by fracturing the non-standard unit, the stick, in the tightest part of the curve 12 times in students' responses to Tasks 3, 4,5, 6B, and 8B. Rose (Grade 6, CRM Group) used this strategy when measuring the spiral-shaped curve for Task 4. She began comparing the curve to the straight object (the stick) by placing half of the stick as a chord inside the tightest part of the curve. She drew a tick mark on the curve to represent the end of half of the stick. Rose then traced along the edge of the stick closest to the curve and re-positioned the stick by rotating it so that most of the second half was a chord inside the curve. Next, she made a tick mark to represent the end of this portion of the stick and traced the along the edge that was closest to the curve. She re-positioned the stick a third time by rotating it so that the small remaining portion was aligned as a chord and made a tick mark to represent the end of this remaining piece.

Next, Rose traced along this small remaining portion of the stick along the edge that was closest to the curve. She continued measuring the rest of the spiral-shaped curve by iterating full stick units. For the final iteration, only slightly more than half of the stick fit. She labeled the third tick mark as one, the fourth tick mark as two, the fifth tick mark as three, the sixth tick mark as four, the seventh tick mark as five, and the eighth tick mark as two thirds (see Figure 58 below).

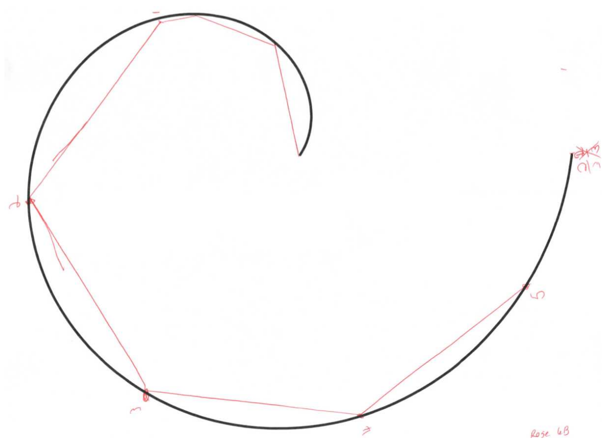


Figure 58. Rose's fracturing of the nonstandard unit in the tightest part of the curve.

Fractured non-standard unit around the entire curve. I observed 20 instances in which students increased precision by fracturing the nonstandard unit, the stick, around the entire curve across Tasks 3, 4, 5, 6B, and 8B. Rick (Grade 8, ICPM Group) used this strategy when comparing the partial circle-shaped curve to a straight object (a stick) for Task 3. He aligned the stick to one end of the curve, placing it as a chord. He allowed the stick to hang over the curve, effectively using only half of the stick as a unit. He placed a finger to keep track of where the stick intersected the curve and iterated the stick, placing the end of the stick at his finger mark each time. When asked to explain how he thought about comparing the curved path to the stick he said, "I would start at the end, and then since if I went like that (showed placing the entire stick as a chord) it'd be more curved so it wouldn't be as long, so I just did halfway (showed by placing half of the stick as a chord). And then I'd do half and half (demonstrated how he iterated the half-stick around the inside of the curve as a chord) and then there's eight halves so four.” Rick exhibited the fractured non-standard unit around the entire curve strategy here by operating on half-stick units as he compared the curve and a stick. I interpreted the observable strategies of fracturing the nonstandard unit in the tightest part of the curve (see Rose’s drawing in Figure 58) and fracturing the non-standard unit around the entire curve (see Rick’s the preceding discussion of Rick) as evidence of coordinating linear extent with another attribute, curve.

Counted a partial unit as a whole. I observed two instances of counting a partial unit as a whole in students’ responses to Tasks 3, 4, 5, 6B, and 8B. Kevin (Grade 4, CLM Group) applied this strategy after using the chord iteration strategy to compare the spiral-shaped curve to the straight object (the stick) for Task 4. He made a record of how he compared the curved path to the stick by first aligning the stick to one end of the curve, placing it as a chord and tracing the edge of the stick closest to the curve. He then repositioned the stick, aligning the endpoint of the stick to the intersection of the curve and the line segment representing the position of the first chord stick unit. He repeated this process drawing five full chord stick units; however only a partial stick fit for the final stick unit (see Figure 59 below).

Kevin showed how many stick units longer the curve was by pointing and counting the segments in his drawing, “One (pointed to the second segment), two (pointed to the third segment), three (pointed to the fourth segment); four (pointed to the fifth segment), five (pointed to the partial sixth segment)." He counted the final partial segment as a whole.

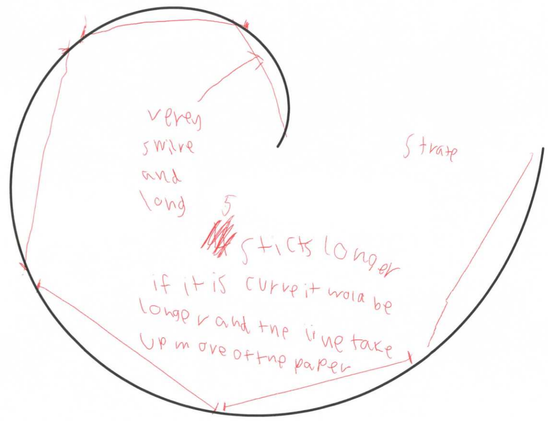


Figure 59. Kevin's counting of a partial unit as a whole.

Compensated for curvature. Students applied a strategy of increasing precision by compensating for curvature a total of 10 times for Tasks 3, 4, 5, 6B, and 8B. Marie (Grade 10, ALM Group) used this strategy when comparing the spiral-shaped curve to a straight object (the stick) for Task 4. She compared the stick to the curve by applying the chord iteration strategy and fracturing the non-standard unit in the tightest part of the curve (see Figure 60 below).

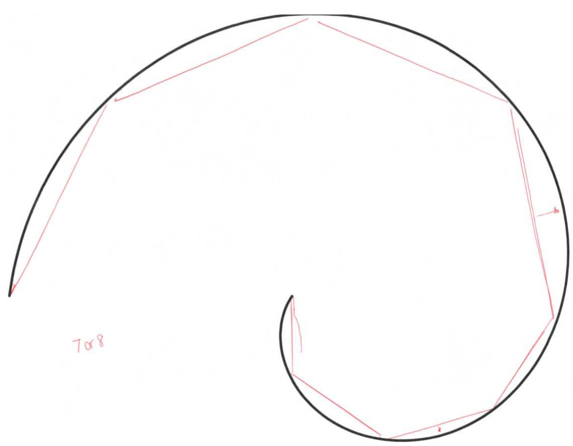


Figure 60. Marie's comparison of a curve and a straight object for Interview Task 4

After measuring Marie said, “OK. So, I when I measured it I got like six and three quarters it looks like. But, again, since it would be pulled up I guess it would be around seven or eight to cover. It would be a little bit more since the curves like here would pull a little bit more in some spots than others.” Marie rounded the number of stick units needed to span the length of the curve from six and three quarters, which she obtained by directly measuring the curve, to seven or eight. That is, she compensated for curvature.

Applied benchmark. I observed two instances of applying a benchmark measurement across Tasks 3, 4, 5, 6B, and 8B. Trent (Grade 6, CRM Group) applied this benchmark strategy when comparing the partial circle-shaped curve and a straight object (the stick) for Task 3. He first placed the stick at the eight and a half inch side of the paper. Next, he drew a tick mark to represent the end of the stick and iterated the stick, aligning the end of the stick with this tick mark and drawing another tick mark at the end of the stick to keep track of the position of this second stick unit. When asked what he was thinking he said, “an average sheet of computer paper's about eight and a half inches long, so this took about two...two times it would be about four and a half inches.” Although Trent’s calculation of half of eight and a half as four and a half was incorrect, he remembered the length of a standard piece of paper in inches and used this information to determine the length of the stick in inches. For him, the length of the short side of a standard sheet of computer paper as eight and a half inches was a benchmark.

Applied conceptual standard unit. The strategy of comparing a straight object to a curve by applying a conceptual standard unit was observed five times in students’ responses to Tasks 3, 4, 5, 6B, and 8B. David (Grade 8, ICPM Group) applied a conceptual standard unit, a mental image of an inch, when comparing the partial circle-shaped curve to a straight object (a stick) for Task 8B:

> David: This is roughly like, three inches...and...um...this is like probably the radius of this (pointed to the curve).
>
> Interviewer: Can you show me how you thought about that?
>
> David: How I thought about how this was the radius?
>
> Interviewer: Yeah.
>
> David: Cuz if you put that there (placed the stick as a radius again) that's like almost half-way from one end of the circle to the other. And then if you would be finding the...um...how long this is (pointed to the curve), which would be like the circumference of it minus that (pointed to the missing part of the circle), which is like a third of it. Um...So you would just find the circumference of the circle and divided it by three.

David then calculated the length of the partial circle-shaped curve in inches.

Intuitions Embedded in Analytical Strategies

Two distinct strategies for comparing a curve and a straight object or using a straight object to compare two curves indirectly involved an intuition, the compression intuition, embedded in an analytical strategy. These strategies are related to either the chord iteration strategy or the tangent iteration strategy.

Tangent curved unit iteration strategy. Participants used the tangent curved unit iteration strategy a total of four times throughout Tasks 3, 4, 5, 6B, and 8B. Ned (Grade 6, CRM Group) exhibited the tangent curved unit iteration strategy when comparing a spiral-shaped curve and a stick for Task 6B. He first placed the stick on the outside of the curve as a tangent aligned with one end of the curve. Next, Ned drew a tick mark at the end of this first stick unit interval and realigned the stick with this tick mark, placing it again as a tangent. He then drew a second tick mark to indicate the end of this second stick unit interval. For the third and fourth iterations of the stick along the tightest part of the curve, he placed the stick as a tangent, aligned with the tick mark representing the end of the previous stick unit interval and allowed the stick to extend beyond the curve. He then drew a tick mark further along the curve than the point at which the stick departed from the curve. A small part of the curve was still extending beyond the fourth iteration. Ned placed part of the stick along this small part of the curve and wrote four and one third sticks. He then explained how he thought about using the stick to help him check saying, "I laid the stick by trying to line it up as...at about as straight as it can go (laid the stick as a tangent on the outside of the spiral-shaped curve) against the line and then...I figured out since it was curving, I would try to straighten it out and then figure out about where it would be if it was straight." Ned's explanation of "figure out about where it would be if it was straight" indicates that he was imagining mentally straightening parts of the curve, at least for the third and fourth tangent stick unit iterations. This suggests that he evoked the compression intuition while using an analytical strategy of directly comparing a stick as a tangent to a curvilinear path.

Other students used a slightly different version of the tangent curved unit iteration strategy. One such example is Grant's (Grade 6, ICPM Group) strategy for comparing a partial circle-shaped curve to a straight object for Task 3. He initially placed the stick on the outside of the curve, aligned with one endpoint. Although the stick extended far beyond the point at which it intersected the curve, he drew a tick mark to represent the end of the stick on the curve. He then repositioned the stick, again placing it as a tangent on the outside of the curve and aligned to the tick mark he had just drawn. Grant again allowed the stick to extend beyond the curve, and he again made a tick mark on the curve beyond where the stick intersected the curve. He repeated this strategy all the way around the curve; the curve extended a small amount beyond the final iteration of the stick. He said it was “Probably like just over four sticks.”

When asked to explain how he thought about comparing the curved path to the stick Grant said, “I just like put the stick where it was (again placed the stick as a tangent aligned with one endpoint of the curve, allowing the stick to extend beyond the point at which it intersected the curve) and I just guessed where it would be if the stick curved.” Grant’s discussion of imagining the stick as curved to guide the placement of his tick marks suggests that he used the compression intuition while applying an analytical strategy of directly measuring the curve with tangent stick units. Although Ned and Grant both applied this tangent curved unit iteration strategy, Ned thought about mentally straightening segments of the curve to match the straight unit and Grant thought about mentally curving the unit to match the curve.

Chord curved unit iteration strategy. Fifteen instances of the chord curved unit iteration strategy appeared in students’ responses to Tasks 3, 4, 5, 6B, and 8B. Kyle (Grade 10, ICPM Group) used the chord curved unit iteration strategy when comparing an S-shaped curve and a straight object in Task 5. He initially placed the stick along one end of the S-shaped curve and drew a tick mark on the curve just before the end of the stick. He then iterated the stick, again placing it as a chord and aligning it with this tick mark. Kyle once again drew a tick mark just before the end of the stick. He continued this process of iterating the stick and drawing tick marks just before the end of the stick the entire way around the S-shaped curve. When asked how he thought about comparing the length of the stick to the length of the curved path Kyle said:

> I…had to take into account that...um...when I put the stick up the curved parts of the line I would end up having to straighten the line out and I would do it with my pen to see how...to make sure that the amount of curved line that I chose would come out to be closest to the length of the stick…so I would just flatten that out and it would come out to here (placed stick as a chord in the position of the second stick unit).

Kyle's explanation about flattening out the curve to match the straight unit suggests that he mentally straightened parts of the curve. This is evidence that he evoked the compression intuition as he applied an analytical strategy of directly measuring the length of the curve in chord stick units. Therefore, the curved chord unit iteration strategy is another example of an intuition embedded in an analytical strategy.

Other students used a different version of the chord curved unit iteration strategy. For example, Rose (Grade 6, CRM Group) compared the stick to the spiral-shaped curved path (Task 6B) by exhibiting a strategy similar to the chord iteration strategy. She placed the stick as a chord aligned with one end of the spiral curve and traced along the edge of the stick closest to the curve. She repositioned the stick at the intersection of the segment representing the initial position of the stick and the curve and again traced along the stick. She repeated this process for four full stick units and one partial stick unit in the tightest part of the curve. When asked how she thought about comparing the curve to the stick she explained, "I imagined if the stick...kind of...was bent. If it were like jello or something I could bend it and then it would fit to there (indicated on the curve how the segment representing the third stick unit would fit along the curve), and then if I could move this, it would stretch out and might do that (traced finger around the piece of the curve spanned by the chord representing the third stick unit)."

Rose's explanation of imagining the stick as bent suggests that she mentally curved the stick, which is evidence that she evoked the compression intuition. Her application of the stick as a unit placed as a chord along the curve did not appear to be influenced by her application of the compression intuition. However, her explanation suggests that the compression intuition was present in her thinking as she applied the analytical chord iteration strategy. Therefore, this is another instance of an intuition, the compression intuition, embedded in an analytical strategy.

Intuitions Used when Comparing Curvilinear Paths to Straight Objects

I observed three of the five intuitions students used to compare curves (Tasks 6A, 7, and 8A) as students compared a curve and a straight object (Tasks 3, 4, and 5) or indirectly compared two curves with a straight object (Tasks 6B and 8B): the compression, straightness, and curve tightness intuitions. Forty-nine responses reflected the use of compression, and 44 of these instances included an analytical strategy.

The compression intuition was used with seven of the analytical strategies observed as students compared curvilinear paths with a straight object (Tasks 3, 4, 5, 6B, and 8B). I observed this intuition most often with the chord iteration and chord curved unit iteration strategies. The straightness intuition occurred only with the chord and tangent iteration strategies. Participants used the curve tightness intuition only with the chord iteration and mixed unit iteration strategies. Only the straightness intuition appeared without an analytical strategy (five instances) when comparing a curvilinear path to a straight object. Table 11 shows how students used intuitions with analytical strategies.

Table 11

Intuitions Used with Analytical Strategies When Comparing a Curve to a Straight Object (Tasks 3, 4, 5, 6B, and 8B)

	Compression	Straightness	Curve Tightness	Analytical without Intuition
Chord Iteration	38.78% (19)	71.43% (5)	80% (4)	16
Continuous Comparison	6.12% (3)	0.00% (0)	0.00% (0)	0
Tangent Iteration	10.20% (5)	28.57% (2)	0.00% (0)	5
Tangent Curved	4.08% (2)	0.00% (0)	0.00% (0)	2
Chord Curved	22.45% (11)	0.00% (0)	0.00% (0)	0
Mixed Unit Iteration	6.12% (3)	0.00% (0)	20% (1)	3
Path Intersection	2.04% (1)	0.00% (0)	0.00% (0)	1
Adjusting Tangency	0.00% (0)	0.00% (0)	0.00% (0)	8
Modified Formula	0.00% (0)	0.00% (0)	0.00% (0)	5
Intuition without analytical	10.20% (5)	0.00% (0)	0.00% (0)	0
Totals	49	7	5	40

* 38.78% of the instances in which the compression intuition was used with an analytical strategy occurred when the chord iteration strategy was used (Tasks 3, 4, 5, 6B, and 8B).

I did not observe the continuous comparison or the chord curved unit analytical strategies without an intuition. Furthermore, participants did not use the adjusting point of tangency and modified circumference formula strategies with an intuition.

Reflecting on Error

For each of the tasks involving comparing curvilinear paths to a straight object (Tasks 3, 4, 5, 6B, and 8B), after making a quantitative comparison, I further probed students' intuitive and analytical thinking about curves by asking *whether* they thought they had over- or underestimated and *how* they knew they had over- or underestimated. Across these five tasks, the 16 interview participants answered that they had underestimated 36 times, overestimated 22 times, and claimed that an answer was neither an over- or underestimate 12 times. Table 12 illustrates the interaction between students' claims about over- or underestimating and the analytical iteration strategies they used to compare a straight object to a curve.

I categorized combinations of these codes related to students' claims about over- or underestimating as correct or incorrect. Two strategies, the chord iteration strategy and the tangent iteration strategy, provided either a clear over- or underestimate. Therefore, a student who used the chord iteration strategy and claimed his or her answer was an overestimate or said he or she did not over- or underestimate was coded as "incorrect acknowledgement of over- or underestimate." Another student who used the same chord iteration strategy, but claimed his or her answer was an underestimate was coded as "correct acknowledgement of over- or underestimate." Some strategies, such as the path intersection iteration strategy in which students attempted to control error by placing the straight object directly on the curve, did not provide clear over- or underestimates. Therefore, the combination of this strategy with an acknowledgement of an over- or underestimate or a claim that an answer was neither an over- nor an underestimate was considered to be neither correct nor incorrect. In Table 12, correct responses are indicated as green, incorrect responses are indicated as red, and responses that could not be clearly identified as correct or incorrect are indicated with grey shading.

Table 12

Interaction Between Iteration Strategies and Statements about Over- or Underestimating when Comparing a Straight Object and a Curve (Tasks 3, 4, 5, 6B, and 8B)

	Acknowledged Underestimate	Acknowledged Overestimate	Claimed answer not an under- or overestimate
Chord Iteration	25	6	3
Continuous Comparison	1	1	0
Tangent Iteration	3	5	1
Tangent Curved	0	4	0
Chord Curved	4	3	2
Mixed Unit	1	3	1
Path Intersection	2	1	0
Adjusting Tangency	2	2	4
Circumference Formula	0	0	2

Students' Justifications when Reflecting on Error

After students' claims about *whether* they had over- or underestimated were analyzed for correctness for Tasks 3, 4, 5, 6B, and 8B, their justifications regarding *why* they thought they had over- or underestimated were analyzed in terms of the intuitions and analytical strategies for curvilinear paths as described in the sections above. Specifically, I coded students' responses according to the five main intuitions for comparing curvilinear paths, the analytical strategies for comparing curves and straight objects, and the analytical strategies related to unit. This analysis focused on the 43 responses that could be clearly defined as correct or incorrect, or those that were associated with the chord and tangent unit iteration strategies.

Participants justified their claims regarding why they thought they had over- or underestimated using an intuition or a combination of intuitions on 18 instances, by discussing an analytical strategy for comparing curves to straight objects on 19 instances, and by discussing an analytical strategy related to unit on 10 occasions. In the following sections I illustrate how students used each intuitive or analytical strategy to justify why they thought they had over- or underestimated when comparing a curve and a straight object for Tasks 3, 4, 5, 6B, and 8B.

Using intuitions to justify claims to reflect on error. Students exhibited seven instances of the straightness intuition, 18 instances of the compression intuition, and 2 instances of the curve tightness intuition when justifying their claims about over- or underestimating when comparing a curve to a straight object. Seven of these instances of intuition use occurred as intuitions used in combination. Mia (Grade 4, CLM Group) used an intuition when reflecting on her way of comparing a straight object to a curve by iterating the stick around the inside of two partial circle-shaped curves as chords, using the whole stick as the unit, for Task 8B. She said the tighter curve was four stick units long and the wider curve was three stick units long. However, she used an intuition to defend claims about both curves being the same length and about overestimating when comparing. She said, “hmmm...Cuz if you curve this one in more (gestured as if to bend the wider curve into the same shape as the tighter curve) then it would look like this one (pointed to the tighter curve).”

Although Mia had measured the tighter curve as four stick units and the wider curve as three stick units, she justified her response by evoking the compression intuition by mentally bending the wider curve into the same shape as the tighter curve. Her conclusion that both curves were the same length, based on the compression intuition, was incorrect. Therefore, I further probed Mia’s thinking about why she thought the curves were the same length, even after determining they were different by directly measuring each curve with the stick:

> Interviewer: OK. And did the stick help you know for sure that they were the same?
>
> Mia: (picked up the stick) A little bit.
>
> Interviewer: A little bit? How so?
>
> Mia: Because this one (pointed to the tighter curve) has four (pointed to the middle of each of the four chord segments representing the length of the tighter curve)…and this one just has three (pointed to the wider curve)…but this one is off (pointed to an error segment created by her drawing of chord stick units on the

tighter curve), so I think that they would still be about the same (pointed to both curves).

Mia's response suggests that she thought that more of the larger error segments on the tighter curve meant that her answer of four stick units was an overestimate. When asked what she would say to convince someone that the curves were the same length she said, "I think that they're the same because this line isn't very exact I think it would be...there would be a lot of lack in there (pointed to an error segment on the tighter curve)...like not very exactly on it, not exactly right…I think if it was exactly on it, there would be less than four." Mia's use of compression when reflecting on the error involved with comparing a curve and a straight object, or the indirect comparison of two curves with a straight object in the case of this particular task, lead her to conclude incorrectly that she had overestimated when using the chord iteration strategy.

Jenny (Grade 4, CLM Group) used an intuition when discussing her comparison between a straight object to a curve for Task 3. She used the chord iteration strategy, using the whole stick as the unit, and answered that the curve was four times longer than the stick. When asked whether she thought the length of the curve was more or less than four, she explained that it was more "because it's curved (traced around the curve with her finger)." Jenny's response reflects an intuition about the curve as being longer than its representation of the curve as four chord stick unit segments, which is consistent with the compression intuition. Unlike Mia, Jenny's application of the compression intuition when reflecting on the error involved with comparing a straight object and a curve resulted in a correct response.

Analytical strategies for comparing to reflect on error. After Jenny used the compression intuition to defend her claim that her comparison between the curve and the straight object for Task 3 involved an underestimate, she also attended to the analytical strategy that she had used to make the comparison. She elaborated on her statements about the length of the curve as being more than the four chord stick units she had measured because it was curved by saying, "and...um...I did this (placed the stick inside the curve as a chord and aligned with one endpoint of the curve)." Therefore, Jenny correctly attended to her strategy of comparing the curve and the stick by iterating the stick around the curve as a chord as a source of error, resulting in an underestimate.

Like Jenny, when Kevin (Grade 4, CLM Group) reflected on whether he had over- or underestimated for Task 6B when comparing a curve and a straight object, he also attended to his analytical strategy of iterating the whole stick as a chord. When asked if he had over- or underestimated, he said he thought he got it "exactly right." Kevin defended this claim by explaining, "Because I tried to line up like the stick almost perfectly on the line and it pretty much turned out like that all the time." Although he had underestimated when comparing the straight object to the curve by representing the curve in chord stick units, he reasoned that his analytical strategy for comparing the straight object to the curve resulted in him lining the stick up perfectly to the line and giving an exactly right comparison. Although Kevin used the same kind of reasoning that Jenny did when reflecting on error, attending to the analytical strategy for comparing a curve and a straight object, his conclusion was incorrect.

Analytical strategies for operating on units to reflect on error. Some students attended to the analytical strategies they used for operating on units (such as mentally curving units, mentally straightening parts of the curve, accuracy of fracturing units, or the alignment of the final stick unit and the end of the curve) to reflect on the error in their comparisons between a straight object and a curve. For example, after using the tangent iteration strategy to compare a straight object and a curve for Task 4, Lynn (Grade 8, CRM Group) correctly stated that she thought her comparison involved an overestimate. She explained her reasoning by attending to her accuracy in partitioning a stick unit into thirds saying, "Slightly over, because my fifth ended here (pointed the endpoint of the stick as it was placed the fifth stick unit position)…and that does not look like two thirds. Two thirds would be like there (spanned fingers to surround the remaining part of the curve not covered by the fifth iteration) so I think it's a little over." Lynn's claim about overestimating the length of the curve using the tangent iteration strategy was correct; however, she only attended to her operations on units to defend this claim.

Trent (Grade 6, CRM Group) also attended to his analytical strategy of operating on units when discussing the error of his comparison between a straight object and a curve for Task 3. After using the chord iteration strategy, he claimed that his comparison between the straight object and the curve yielded an overestimate "because this was a little bit longer than the curve (placed the stick in the fourth stick unit position to illustrate that the stick extended beyond the end of the curve)." Like Lynn, Trent attended to his operations on units to defend a claim about the error involved with his comparison of the straight object and the curve. However, the claim Trent was defending, that he had overestimated using the chord iteration strategy, was incorrect.

Using multiple justifications. Six different students justified their claims using either a combination of two analytical strategies (one related to comparing curves to straight objects and one related to unit) or an intuition and an analytical strategy (either an analytical strategy related to comparing curves to straight objects or one related to unit). Jenny (Grade 4, CLM Group), described above, explained why she thought her comparison between a straight object and a stick for Task 3 was an underestimate using both an intuition and the analytical strategy she used to compare the straight object to the curve. Like Jenny, Trent (Grade 6, CRM Group), Grant (Grade 6, ICPM Group), and Marie (Grade 10, ALM Group) exhibited one instance of using both an intuition with a discussion about the analytical strategy used to compare a straight object to a curve when defending a claim about whether a comparison resulted in an over- or underestimate.

Rick (Grade 8, ICPM Group) justified why he thought his comparison of a curve and a straight object for Task 6B was an underestimate by using a combination of two analytical strategies: one related to his way of comparing a curve to a straight object and another related to his way of operating on the unit. After using the chord iteration strategy to compare he said, "I think I underestimated this one (pointed to the curve with the straight segment) more than this one (pointed to the spiral curve) because this one is like…this has more curve to it (points to the tightest part of the curve with the straight segment) I wasn't dead on the line...And I didn't even get all the way to the end either…" His discussion about not being "dead on the line" indicates that he was reflecting on the fact that he used an analytical strategy of comparing the straight object to the curve by representing the straight object as chord stick units on the curve, resulting in an underestimate. Rick elaborated by also mentioning his operations on units. He noted that he "didn't even get all the way to the end either," indicating an attention to the curve extending beyond the final chord stick unit. Rick's claim about underestimating when comparing the straight object and the curve was correct.

Kevin (Grade 4, CLM Group) defended his claim that his comparison of a curve to a straight object for Task 8B was an overestimate using an intuition and an analytical strategy related to his way of comparing the curve to the straight object. He used the chord iteration strategy to compare the straight object to each curve. When asked whether he thought he had over- or underestimated when comparing he said, "I think I overestimated a little, because this one it goes off a little (pointed to the end of the wider curve where the third stick unit extended beyond the curve), and this one when it was going here it was more of just like a straight line (pointed to the first stick unit for the tighter curve). The stick can't like curve into a line like a circle." Kevin's discussion of curving the stick indicates that he evoked the compression intuition. He also discussed his operations on units; he attended to a stick unit extending beyond the curve to defend his claim that he overestimated. Kevin evoked both an intuition and an analytical strategy, but his claim that he overestimated when using the chord iteration strategy was incorrect.

Relating Correctness and Students' Justifications when Reflecting on Error

Table 13 illustrates the interaction between the correctness of a claim about over- or underestimating and the use of intuitions, analytical strategies for comparing curves and straight objects, and analytical strategies related to unit for justifying claims.

Table 13

Interaction Between Correctness and Students' Justifications when Reflecting on Error When Comparing a Curve to a Straight Object (Tasks 3, 4, 5, 6B, and 8B)

	Use of an Intuition	Analytical strategy for comparing	Analytical strategy for units
Correct acknowledgement of an over- or underestimate	77.78%* (14)	84.21% (16)	40.00% (4)
Incorrect acknowledgement of an over- or underestimate	22.22% (4)	15.79% (3)	60.00% (6)
Totals	18	19	10

* 77.78% of the instances in which an intuition was used to defend whether a comparison between a curve and a straight object was an over- or an underestimate were correct.

Students' discussions about whether a comparison between a curve and a straight object was an over- or an underestimate that involved either an intuition or a discussion about the analytical strategy used to make the comparison yielded a correct answer in most instances, with 77.78% and 84.21% respectively. However, when students attended to the analytical strategy used to operate on units, such as a partitioning of units or the alignment of the final stick unit iteration and the end of the curve, their answer was correct only 40% of the time.

Relating Intuitions and Analytical Strategies for Curvilinear Paths to the Length LT

I tracked patterns in intuition and analytical strategy use for comparing a nonstandard unit and a curve (Tasks 3, 4, 5, 6B, and 8B) within and across the LT groups. I differentiated and categorized analytical strategies and intuitions for comparing and analytical strategies related to unit according to their appearance within and across the LT groups to explore developmental patterns. In the sections below I describe these developmental patterns with respect to the use of intuitions and analytical strategies for comparing as well as the participants' use of analytical strategies related to operating on units.

Figures 61 and 62 illustrate the frequency of each intuition and analytical strategy for comparing a straight object and a curve. In each column, the darkest shade of blue indicates the LT level group with the highest frequency and the lightest shade of blue shows the LT level group with the lowest frequency of a particular intuition or analytical strategy.

Tasks 3, 4, 5, 6B, & 8B: Analytical Strategies Related to Comparing

	Chord Iteration Strategy	Continuous Comparison Strategy to Estimate	Tangent Iteration Strategy	Mixed Unit iteration strategy	Path Intersection Iteration Strategy	Adjusting Point of Tangency Iteration Strategy	Applied Modified Circumference Formula	Tangent Curved Unit Iteration Strategy	Chord Curved Unit Iteration Strategy	Direct Measurement Strategies (without embedded intuition)	Direct Measurement Strategies (with embedded intuition)	Indirect Measurement Strategy
CLM Group	20	2	3	2	2	0	0	0	0	29	0	0
CRM Group	12	0	4	5	0	1	0	1	6	22	7	0
ICPM Group	9	0	4	0	1	7	2	3	2	21	5	2
ALM Group	10	0	5	1	0	3	3	0	7	19	7	3
Totals	51	2	16	8	3	11	5	4	15	91	19	5

Figure 61. Patterns of analytical strategy use for comparing straight objects and curves within and across LT groups.

Tasks 3, 4, 5, 6B, & 8B: Intuitions Related to Comparing

	Compression Intuition	Straightness Intuition	Curve Tightness Intuition	Combination of Intuitions	Total Intution Use
CLM Group	9	2	2	1	13
CRM Group	16	0	1	0	17
ICPM Group	8	3	1	1	12
ALM Group	14	2	1	2	17
Totals	47	7	5	4	59

Figure 62. Patterns of intuition use for comparing straight objects and curves within and across LT groups.

I describe the developmental patterns within and across the four length LT level groups included in the present study, which are depicted in Figures 61 and 62, in the sections below beginning with Table 14.

Table 14

Distribution of Analytical Strategies for Comparing Curves and Straight Objects (Tasks 3, 4, 5, 6B, and 8B) within and across Length LT Level Groups

	CLM Group	CRM Group	ICPM Group	ALM Group	Totals
Chord iteration	39.22%* (20)	23.53% (12)	17.65% (9)	19.61% (10)	51
Continuous comparison	50.00% (2)	0.00% (0)	50.00% (2)	0.00% (0)	4
Tangent iteration	21.43% (3)	28.57% (4)	14.29% (2)	35.71% (5)	14
Mixed unit iteration	25.00% (2)	62.50% (5)	0.00% (0)	12.50% (1)	8
Path intersection iteration	66.67% (2)	0.00% (0)	33.33% (1)	0.00% (0)	3
Adjusting tangency	0.00% (0)	9.10% (1)	63.63% (7)	27.27% (3)	11
Tangent curved unit	0.00% (0)	25.00% (1)	75.00% (3)	0.00% (0)	4
Chord curved unit	0.00% (0)	40.00% (6)	13.33% (2)	46.67% (7)	15
Modified formula	0.00% (0)	0.00% (0)	40.00% (2)	60.00% (3)	5

* 39.22% of the instances in which the chord iteration strategy was used occurred when CLM students compared a straight object and a curve for a total of 20 instances.

Patterns of Analytical Strategy Use within LT Groups

Participants within each length LT level group used the chord iteration strategy most often. For the CLM group, this was followed by the use of four additional analytical strategies: the continuous comparison and tangent, mixed unit, and the path intersection iteration strategies. Within the CRM group, this was followed by three more analytical strategies: mixed unit, tangent, adjusting point of tangency iteration strategies. At the CRM level, students used two analytical strategies with embedded intuitions: chord curved unit and tangent curved unit iteration strategies. The group of ICPM level participants used all but one of the analytical strategies observed for comparing a curve and a straight object, the mixed unit iteration strategy. The students in the ALM level group used all of the analytical strategies except the path intersection and tangent curved unit iteration strategies.

Patterns of Analytical Strategy Use across LT Groups

I observed the chord iteration strategy most often in the CLM level participants' responses, and the use of this strategy generally decreased as the levels of the LT for length measurement increased in sophistication. Few instances occurred for most analytical strategies for comparing a curve and a straight object shown in Table 14. Therefore, I organized codes according to three thematic categories. The direct measurement category includes the continuous comparison and chord, tangent, mixed unit, path intersection, and adjusting point of tangency iteration strategies. The direct measurement with embedded intuition use strategies include the tangent curved unit and chord curved unit iteration strategies. The indirect measurement category consists of the modified circumference formula strategy. In Table 15, I describe the interaction of the thematic categories for the analytical strategies observed in the present study within and across groups.

Table 15

Interaction of Thematic Categories for Analytical Strategies for Comparing Curves and Straight Objects (Tasks 3, 4, 5, 6B, and 8B) within and across Length LT Level Groups

	CLM Group	CRM Group	ICPM Group	ALM Group	Totals
Direct measurement: analytical without embedded intuition	31.78%* (29)	24.18% (22)	23.08% (21)	21.88% (19)	91
Direct measurement: analytical with embedded intuition	0.00% (0)	36.84% (7)	26.32% (5)	36.84% (7)	19
Indirect measurement	0.00% (0)	0.00% (0)	40% (2)	60% (3)	5

* 31.78% of the instances in which a direct measurement strategy was observed as students compared a straight object to a curve occurred in the CLM level group.

Patterns in Thematic Categories of Analytical Strategy Use within LT Groups

Within the CLM level group, only direct measurement analytical strategies without embedded intuitions were used. At the CRM level, students used only direct measurement strategies, with or without embedded intuitions. For the ICPM and ALM level groups, I observed direct measurement strategies with or without embedded intuitions and a strategy for indirect measurement (applying a modified circumference formula).

Patterns in Thematic Categories of Analytical Strategy Use across LT Groups

Table 15 shows that the analytical direct measurement strategies were used most often by students in the CLM group, the lowest group, and decreased across the groups as the levels increased in sophistication. Analytical direct measurement strategies with embedded intuitions were not observed in the CLM group and were almost evenly distributed across the CRM, ICPM, and ALM groups. The indirect measurement strategy, a modified circumference formula, was observed only in the ICPM and ALM groups.

I tracked patterns of analytical strategy use related to unit when comparing straight objects and curves (Tasks 3, 4, 5, 6B, and 8B) within and across the groups representing CLM, CRM, ICPM, and ALM levels of the length LT. The number of instances of each analytical strategy related to unit for comparing a curve and a straight object is shown in Figure 63 below. In this figure, within each of the columns, the darkest shade indicates the LT level with the highest frequency of a specific analytical strategy, and white indicates the LT level with the lowest frequency of a specific analytical strategy related to unit.

Tasks 3, 4, 5, 6B, & 8B: Analytical Strategies Related to Unit

	Used the Whole Stick as a Unit	Fractured Unit Once	Fractured Unit in the Tightest Part of the Curve	Fractured Unit along Entire Curve	Counted a Partial Unit as a Whole Unit	Compensated for Curvature	Mentally Curved Straight Unit	Mentally Straightened Part of the Curve	Applied Conceptual Standard Unit	Applied Benchmark	Fractured Unit	Applied Mental Units	Mentally Transformed the Unit or the Curve
CLM Group	25	7	1	2	1	0	0	0	0	0	10	0	0
CRM Group	23	10	2	5	1	4	1	5	4	2	17	6	6
ICPM Group	14	6	5	8	0	0	4	0	1	0	19	1	4
ALM Group	13	8	4	5	0	6	0	7	0	0	17	0	7
Totals	75	31	12	20	2	10	5	12	5	2	63	7	17

Figure 63. Patterns of analytical strategy use related to unit for comparing straight objects and curves within and across LT groups.

Figure 63 shows developmental patterns for analytical strategy use related to unit across LT groups. To explore developmental patterns in analytical strategy use related to unit across the LT levels, codes related to conceptually congruent themes concerning the analytical strategies related to unit (Tasks 3, 4, 5, 6B, and 8B) were collapsed. Codes describing strategies for fracturing units (fractured nonstandard unit once at the endpoint of the curve, fractured nonstandard unit in the tightest part of the curve, and fractured nonstandard unit around the entire curve), mentally transforming the unit or the curve (mentally curved unit and mentally straightened curve segments), and the application of mental units (applied benchmark and applied conceptual standard unit) were collapsed.

Analytical strategies mentally curved straight unit and mentally straightened curve segments occurred with the chord curved and tangent curved unit strategies. Table 16 shows how the distribution of these collapsed codes relate to the LT levels.

Table 16

Distribution of Conceptually Congruent Analytical Strategies Related to Unit (Tasks 3, 4, 5, 6B, and 8B) across Length LT Groups

	CLM Group	CRM Group	ICPM Group	ALM Group	Totals
Used the whole stick as a unit	33.33%* (25)	30.67% (23)	18.67% (14)	17.33% (13)	75
Fractured unit	15.87% (10)	26.99% (17)	30.16% (19)	26.99% (17)	63
Fractured unit once	22.58% (7)	32.36% (10)	19.35% (6)	25.81% (8)	31
Fractured unit in tightest part of curve	8.33% (1)	16.67% (2)	41.67% (5)	33.33% (4)	12
Fractured unit along entire curve	10.00% (2)	25.00% (5)	40.00% (8)	25.00% (5)	20
Counted partial unit as whole	50.00% (1)	50.00% (1)	0.00% (0)	0.00% (0)	2
Compensated for curvature	0.00% (0)	40.00% (4)	0.00% (0)	60.00% (6)	10
Applied mental units	0.00% (0)	85.71% (6)	14.29% (1)	0.00% (0)	7
Mentally transformed the unit or curve	0.00% (0)	35.29% (6)	23.53% (4)	41.18% (7)	12

* 33.33% of the instances in which the whole stick was used as a unit were observed in students from the CLM group when comparing a curve and a straight object.

Patterns of Analytical Strategy Use Related to Unit within LT Level Groups

Within the CLM level group, the strategy of using the whole stick as a unit was observed most often. This was followed by instances of fracturing the unit once, and one instance each of fracturing the unit in the tightest part of the curve and counting a partial unit as a whole unit. CRM level students also exhibited the strategy of using the whole stick as the unit most often. This was followed by instances of fracturing the unit once, applying mental units, and mentally transforming the unit or curve. Few instances of fracturing the unit in the tightest part of the curve or along the entire curve, counting a partial unit as a whole, and compensating for curvature were also observed in the CRM level group. For the ICPM and ALM level participants, instances of fracturing units occurred more often than instances of using the whole stick as the unit. At both of these levels, several instances each of fracturing the unit once, fracturing the unit in the tightest part of the curve, and fracturing the unit along the entire curve were all observed. Within the ICPM level, participants used the strategy of applying the whole stick as a unit and mentally transforming the unit or curve; however, only one instance of applying mental units was observed. At the ALM level, instances of using the whole stick as the unit, compensating for curvature, and mentally transforming the unit or curve occurred.

Patterns of Analytical Strategy Use Related to Unit across LT Level Groups

Table 16 indicates that the instances of the analytical strategy of using the whole stick as the unit occurred most often within the lowest level group included in the study, the CLM level group, with 25 occurrences. The table also illustrates a trend of decreasing instances of using the whole stick as the unit as the level groups increased in sophistication, across the CRM, ICPM, and ALM levels. Overall, the fewest instances of fracturing units occurred within the CLM group, and there exists an overall trend of increasing instances of fracturing units as the level groups increase in sophistication. Within the CLM and CRM level groups, participants mainly fractured units when a whole unit could not fit at the end of the curve. By the ICPM and ALM levels, participants exhibited more instances of fracturing units in the tightest part of the curve and along the entire curve. I interpreted this observable strategy of fracturing units in the tightest part of the curve or along the entire curve as evidence of coordinating linear extent with curve. Therefore, this suggests that, by the ICPM level, students were able to coordinate linear extent with curve. Counting partial units as whole units occurred only at the CLM and CRM groups, whereas the application of mental units and mentally transforming the curve or the unit occurred only within the CRM, ICPM, and ALM level groups. The strategy of comparing a curve and a straight object by applying mental units (either a benchmark or a conceptual standard unit) occurred most often at the CRM level.

Reflecting on Error

I tracked students' statements about the error involved in their comparisons between a straight object and a curve (Tasks 3, 4, 5, 6B, and 8B) within and across the LT groups. Patterns in the ways participants reflected on error involved are illustrated in Figure 64. In this figure, the darkest shade indicates the LT level group with the highest frequency of a particular code related to students' reflections on the error involved.

Tasks 3, 4, 5, 6B, and 8B: Reflecting on Error

	Acknowledged Underestimate	Acknowledged Overestimate	Claimed Answer Was Not an Over- or Underestimate	Incorrectly Acknowledged Over- or Underestimate	Correctly Acknowledged Over- or Underestimate	Acknowledgement of Over- or Underestimate was Neither Correct nor Incorrect
CLM Group	13	6	1	3	14	6
CRM Group	7	6	2	5	5	6
ICPM Group	9	7	3	2	5	9
ALM Group	7	3	6	2	6	4
Totals	36	21	12	12	30	25

Figure 64. Patterns of students' statements regarding the error involved in their comparisons between straight objects and curves within and across LT groups.

Figure 64 illustrates developmental patterns for whether students thought they had over- or underestimated and whether their statements about over- or underestimating were correct. In the sections below I describe these patterns, starting with Table 17, which illustrates the interaction between LT groups and claims about over- and underestimating.

Table 17

Distribution of Claims of Over- and Underestimates When Reflecting on Error (Tasks 3, 4, 5, 6B, and 8B) across Length LT Level Groups

	CLM Group	CRM Group	ICPM Group	ALM Group	Totals
Acknowledged Underestimate	36.11%* (13)	19.44% (7)	25.00% (9)	19.44% (7)	36
Acknowledged Overestimate	27.27% (6)	27.27% (6)	31.82% (7)	13.64% (3)	22
Claimed answer was not an under- or overestimate	8.33% (1)	16.67% (2)	25.00% (3)	50.00% (6)	12
Totals	28.57% (20)	21.43% (15)	27.42% (19)	22.86% (16)	70

* 36.11% of the instances in which a student acknowledged an underestimate when comparing a straight object and a curve occurred in the CLM group.

Patterns in Claims of Over- and Underestimates within LT Level Groups

In each LT group, participants claimed they underestimated most often. This was followed by acknowledgements of overestimates and claims that a comparison was neither an over- nor an underestimate. This same distribution was observed in the CLM, CRM, and ICPM groups. However, for ALM, claims that a comparison did not involve an over- or underestimate occurred more often than acknowledgements of overestimates.

Patterns in Claims of Over- and Underestimates across LT Level Groups

Table 18 shows that acknowledgements of underestimates when comparing a straight object and a curve occurred most often in the CLM level group. I observed acknowledgements of overestimates with approximately the same frequency across the CLM, CRM, and ICPM level groups; however, such claims decreased for the ALM level group. The number of claims that an answer was neither an over- nor underestimate was at a minimum for the CLM level and increased in frequency as the LT level groups increased in sophistication.

Based on strategies for comparing, a claim that a comparison resulted in an over- or underestimate or neither an over- nor underestimate may have been correct or incorrect. Table 18 illustrates the interaction between the correctness and the levels of the LT for length measurement.

Table 18

Distribution of Correctness for Claims of Over- and Underestimates When Reflecting on Error (Tasks 3, 4, 5, 6B, and 8B) across Length LT Level Groups

	CLM Group	CRM Group	ICPM Group	ALM Group	Totals
Incorrectly acknowledged over- or underestimate	13.04%* (3)	31.25% (5)	12.5% (2)	8.33% (2)	12
Correctly acknowledge over- or underestimate	60.89% (14)	31.25% (5)	31.25% (5)	50.00% (6)	30
Acknowledgement of over- or underestimate was neither correct nor incorrect	26.09% (6)	37.5% (6)	56.25% (9)	33.33% (4)	25
Totals	23	16	16	12	67

* CLM students' discussions about comparing a straight object and a curve resulted in an incorrect acknowledgement of an over- or underestimate 13.04% of the time.

Patterns in Correctness when Reflecting on Error within LT Groups

Participants in the CLM and ALM level groups discussed their comparisons between a straight object and a curve by correctly citing an over- or underestimate most often. For both of these LT level groups, this was followed by claims that an over- or underestimate was neither correct nor incorrect and incorrectly acknowledging an over- or underestimate. For the CRM and ICPM level groups, participants' acknowledgements of over- or underestimates were neither correct nor incorrect most often. This was followed by correct acknowledgements of over- or underestimates. Within the CRM group, these correct statements occurred with the same frequency as incorrect statements about over- or underestimates. At the ICPM level, incorrect statements about over- or underestimates occurred least often.

Patterns in Correctness when Reflecting on Error across LT Groups

Participants in the CLM level group exhibited the highest frequency of acknowledgements of over- or underestimates. The highest number of instances of acknowledgements of over- or underestimates that could not be considered as correct or incorrect occurred within the ICPM level group. The largest number of instances of incorrectly acknowledging an over- or underestimate was observed at the CRM level. The frequency of instances within each category in Table 18 was evenly distributed for the CRM level group.

I tracked students' justifications for why they thought they had over- or underestimated when comparing a curve to a straight object within and across the four length LT level groups. Table 19 illustrates the distribution of students' justifications across the LT level groups.

Table 19

Distribution of Intuitions and Analytical Strategies for Reflecting on Error (Tasks 3, 4, 5, 6B, and 8B) across Length LT Level Groups

	CLM Group	CRM Group	ICPM Group	ALM Group	Totals
Intuition	40.00%* (8)	25.00% (2)	60.00% (6)	22.22% (2)	18
Analytical strategy for comparing a straight object and curve	40.00%* (8)	12.50% (1)	20.00% (2)	66.67% (6)	19
Analytical strategy for unit operation	10.00% (2)	62.50% (5)	20.00% (2)	11.11% (1)	10
Totals	20	8	10	9	47

* CLM level students' discussions about the error when comparing a straight object and a curve reflected the use of an intuition for 40% of the responses given by CLM level students.

Intuition and Analytical Strategy Use for Reflecting on Error within LT Groups

Table 19 indicates that CLM level participants relied on justifying their claim by discussing analytical strategies they had used to compare a straight object and a curve or intuitions, with the same frequency. At the CRM level, students relied most often on justifying claims by discussing the analytical strategies used to operate on units, such as fracturing units. Within the ICPM level group, students most often relied on intuitions. At the highest level, the ALM level, participants most often discussed their analytical strategies for comparing a straight object and a curve.

Intuition and Analytical Strategy Use for Reflecting on Error across LT Groups

Clear developmental patterns were not observed in participants' justifications of claims about why they had over- or underestimated. The use of intuition occurred most often at the CLM and ICPM levels. Attention to analytical strategies for comparing a straight object and a curve occurred most often at the CLM and ALM levels. Justifications involving a discussion of analytical strategies for operating on units occurred most often at the CRM level, and were approximately evenly distributed across the CLM, ICPM, and ALM levels.

Analytical and Intuitive Thinking for Measuring Curves

Two tasks were posed for the purpose of probing students' intuitive and analytical strategies for measuring curvilinear paths, Tasks 9 and 10 (Figures 65 and 66 below, respectively). For Tasks 9 and 10 students were provided with an image of a curvilinear path printed on gridded paper, which they were told represents the outline of a fancy doorway on a blueprint. They were then given a standard ruler and asked to measure the outline of the fancy doorway in the most precise possible way.

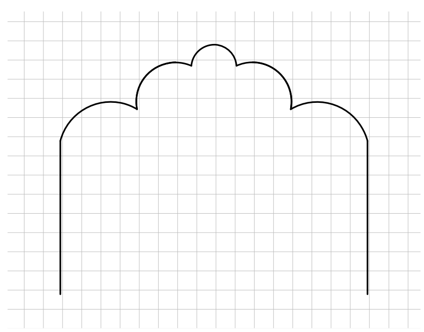


Figure 65. Measuring a curvilinear path, interview Task 9.

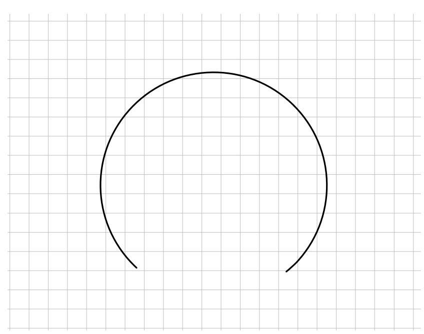


Figure 66. Measuring a curvilinear path, interview Task 10.

I coded students' responses with respect to the intuitive thinking exhibited in their discussions of how they measured the doorway and their analytical thinking indicated by their strategies for measuring the curves with the ruler.

Analytical Strategies for Measuring a Curve with a Ruler

Five of the seven analytical strategies that were observed when students compared a straight object to a curve (Tasks 3, 4, 5, 6B, and 8B) were also observed when students measured a curve with a ruler: chord iteration strategy, tangent iteration strategy, path intersection iteration strategy, adjusting point of tangency iteration strategy, and the modified circumference iteration strategy.

Chord iteration strategy. As was observed when students compared a straight object and a curve (Tasks 3, 4, 5, 6B, and 8B), students who used the chord iteration strategy when measuring a curve with a ruler placed a standard unit (an inch or centimeters), fraction of a standard unit (a quarter or half inch), or composition of standard units (2, 3, or 10 centimeters or 2 inches) as a chord on the interior of the curve. Nine instances of this strategy were observed in five different students' responses to Tasks 9 and 10: Mia (Grade 4, CLM Group), Kevin (Grade 4, CLM Group), Trent (Grade 6, CRM Group), Rick (Grade 8, ICPM Group), and Marie (Grade 10, ALM Group). For example, Trent used the chord iteration strategy to measure the partial circle curve for Task 10 (see Figure 66). He first partitioned the curve into two halves. Then Trent aligned the ruler to the leftmost endpoint of the curve. Next, he used the interval on the ruler from 0 to 1 as a chord and traced along the edge of the ruler closest to the curve to create a 1-inch chord segment. Trent repositioned the ruler with the zero point aligned to the intersection of the first inch segment and the curve and again used the interval from 0 to 1 as a chord to guide the placement of the second chord inch segment. He repeated this procedure around half of the curve, creating seven 1-inch chord segments, with the end of the seventh segment meeting the vertical line he had drawn to partition the curve into two halves. He said, “OK. One, two, three, four, five, six, seven” and wrote 7 x 2 = 14. He then said the length of the curve was “around 14 inches.”

Tangent iteration strategy. Similar to a strategy observed when students compared a straight object and a curve (Tasks 3, 4, 5, 6B, and 8B), students used the tangent iteration strategy to measure a curve with a ruler. There were six instances of the tangent iteration strategy reflected in four students’ responses to Tasks 9 and 10: Noah (Grade 4, CLM Group), Jenny (Grade 4, CLM Group), Trent (Grade 6, CRM Group), and Lynn (Grade 8, CRM Group). For example, Jenny used this strategy when measuring the partial circle curve with the ruler for Task 10 (Figure 66). She initially placed the ruler as a tangent on the outside of the curve aligned with one endpoint. Jenny then used the tick mark on the ruler that was labeled as 1 to guide her drawing of a tick mark on the curve. Next, she repositioned the ruler as a tangent and aligned with the tick mark she had drawn and again used the tick mark on the ruler labeled as 1 to guide the placement of another tick mark on the curve. She continued this process of positioning the ruler as a tangent and applying the interval from 0 to 1 on the ruler to guide her drawing of the next tick mark on the curve. She said the curve was 14 inches.

Path intersection iteration strategy. The path intersection strategy, which was also observed when students compared a straight object to a curve, was reflected in one students’ response for measuring a curve with a ruler. Lynn (Grade 8, ICPM Group) used this strategy when measuring the outline of the doorway for Task 9 (see Figure 66). Lynn first correctly measured one of the straight segments and labeled it as 4 inches. Next, she realigned the ruler so that the tick mark labeled as 1 inch was aligned with a tick mark she drew at the end of the straight segment she had just measured. She placed the ruler directly on this curved segment, drew a tick mark on it, and labeled this section as one inch. She continued this process of realigning the ruler and placing a portion of the ruler direction on the curve, drawing a tick mark, and labeling the section with ½, 1 or 1 ½ inches. When asked how she thought about measuring the curve with the ruler Lynn said:

> Lynn: When I went here (placed the ruler along one of the straight segments), I went as straight as possible. And then here I just tried to go straight around (indicated with her ruler on the curved segments, presumably to show that she used the largest possible interval on the ruler that she could match to a curved portion on a curved segment).
>
> Interviewer: So, you found parts that were straight?
>
> Lynn: Yeah.

Adjusting point of tangency iteration strategy. Students who used the adjusting point of tangency iteration strategy placed the ruler as a tangent to the curve and rotated the ruler, adjusting the point of tangency and accumulating the length of the curve along the ruler. This strategy was also observed when students compared a straight object to a curve (Tasks 3, 4, 5, 6B, and 8B). Four students exhibited a total of seven instances of this strategy when measuring a curve with a ruler (Tasks 9 and 10): Mia (Grade 4, CLM Group), (Grade 6, CRM Group), Ruth (Grade 8, ICPM Group), and Scott (Grade 10, ALM Group).

Mia (Grade 4, CLM Group) used the adjusting point of tangency iteration strategy to measure the curve for Task 9 (Figure 66). Using the centimeter side, she aligned the ruler along one of the straight segments and labeled it as 10. Mia then realigned the ruler to the intersection of the straight segment and the first curved segment. Next, she rotated the ruler around to the intersection of the first curved segment and the second curved segment, adjusting the point of tangency and accumulating the length of the first curved segment on the ruler. She wrote "7" next to the first curved segment. Mia repeated this process of realigning the ruler with the intersection of the previous curved segment and the next curved segment to be measured, rotating the ruler around the outside of the curved segment, accumulating the length of the curved segment on the ruler, and writing the length above the segment. She then added all of the straight and curved segments that she had measured as 10 + 10 + 7 + 7 + 6 + 6 + 5 = 51.

Modified circumference formula strategy. I observed the modified circumference formula in students' responses to tasks involving the comparison of a curve to a straight object (Tasks 3, 4, 5, 6B, and 8B) as well as tasks involving the measurement of a curve with a ruler (Tasks 9 and 10). Students who applied a modified circumference formula strategy used the formula for the circumference of a circle in their solutions. Three instances of this strategy were observed as students measured a curve with a ruler: David (Grade 8, ICPM Group), Zane (Grade 10, ALM Group), and Scott (Grade 10, ALM Group).

David (Grade 8, ICPM Group) applied a modified circumference formula when measuring the partial circle curve for Task 10 (Figure 66). He placed the ruler vertically across the rounded doorway and drew a vertical line. He then drew a horizontal line, which intersected the vertical line in the center of the partial circle curve. Next, he drew two segments from the intersection of the vertical and horizontal lines to each endpoint of the partial circle curve. He multiplied 6.28 by 15 to get an answer of 94.2 and then divided 94.2 by 3 to get 31.4. Finally, he subtracted 31.4 from 94.2 to get 62.8, which he wrote and circled. When asked how he thought about measuring the curve he said, "...I found the circumference of it because it's a circle…and then I divided it by three because that is roughly one third of it (spanned finger across the open part of the circle)."

Intuition embedded in analytical strategies for measuring a curve with a ruler. In addition to the analytical strategies discussed in the section above, I observed the same two types of intuitions embedded in analytical strategies that I saw in students' responses for tasks involving the comparison of a straight object and curve (Tasks 3, 4, 5, 6B, and 8B) and using rulers to measure curves (Tasks 9 and 10). These strategies were the tangent curved unit iteration strategy and the chord curved unit iteration strategy. Similar to the application of these strategies to comparisons between a straight object and a curve, students either mentally straightened parts of the curve to match a section of the ruler (such as an inch or a centimeter) or mentally curved a segment of the ruler to match a part of the curve.

The mental straightening of part of the curve or mental curving of part of the ruler is an illustration of the application of the compression intuition that has been described elsewhere. Therefore, the compression intuition is embedded in both the tangent curved unit and chord curved unit strategies. The compression intuition was the only intuition observed in students' responses to Tasks 9 and 10. This intuition was only observed as the mental straightening of parts of curves or mental curving of part of the ruler, as part of these strategies as students used a ruler to measure a curve (for Tasks 9 and 10).

Tangent curved unit iteration strategy. Students who used the tangent curved unit iteration strategy placed the ruler on the outside of the curve as a tangent and then mentally curved part of the ruler to match a section of the curve or mentally straightened part of the curve to match the ruler. Two instances of this strategy were observed in one student's responses to Tasks 9 and 10, Grant (Grade 6, ICPM Group). For example, when measuring the partial circle curve with the ruler for Task 10 (Figure 66), Grant placed the ruler as a tangent to the curve with the zero point of the ruler aligned to one endpoint. He then drew a tick mark on the curve and labeled it as "3." Next, he realigned the ruler to this tick mark placing it as a tangent and drew another tick mark, which he labeled as "6." He then continued this process around the curve, partitioning the curve into 3-centimeter segments. When asked about his method for measuring the curve in the most precise way that he could, he said:

> Grant: Um…I imagined if the ruler was curved and I marked like every three centimeters.
>
> Interviewer: OK. Could you show me like how you did it, say from here to here (traced finger around the segment of the curve between his tick marks labeled as 3 and 6)?
>
> Grant: I just like put it here (aligned the zero point of the ruler to the tick mark labeled as 3) and like, if it was curved, it would probably go like, right there (showed where the tick mark labeled as 3 on the ruler would intersect with the curve if the ruler was curved).

Interviewer: OK. I think I see it. So, why did you decide three centimeters?

Grant: Uh...cuz it was quicker than two.

Interviewer: I see. And why not four?

Grant: Uh...because four would like...it would be harder to like guess if it was curved with this. It would be like longer measurements so it would be harder to guess where it would be if it was curved.

For each instance in which Grant applied the tangent curved unit iteration strategy when measuring a curve with a ruler, he applied the compression intuition by mentally curving the unit, which was a composite of 3 centimeters in this case.

Chord curved unit iteration strategy. Students who applied the chord curved unit iteration strategy placed the ruler on the inside of the curve as a chord. They then applied the compression intuition by either mentally curving part of the ruler to match the curve or mentally straightening part of the curve to match the ruler. I observed four instances of this strategy in two students' responses as they measured a curve with a ruler: Ned (Grade 6, CRM Group) and Kyle (Grade 10, ALM Group).

For example, Kyle applied the chord curved unit iteration strategy when measuring the partial circle curve with the ruler for Task 10 (Figure 66). He placed the interval from 0 to 1 on the ruler as a chord on the inside of the curve and aligned with one of the endpoints. He then drew a tick mark on the curve, realigned the interval from 0 to 1 on the ruler as a chord on the inside of the curve, and then drew another tick mark. He continued applying the interval from 0 to 1 as a chord around the curve and said:

Kyle: I found that all together, if you were to straighten the whole thing out, it would be 14.25 inches.

Interviewer: OK. Now, tell me a little bit about how you decided to make these marks right here (pointed to the tick marks on the curve)

Kyle: Um...since I was using inches and it's not that much of a far distance apart...and, also, in that amount of space the curve isn't too far going out, I ended up...um...starting at 1 and then ending right about there (pointed to a mark near the zero point). So, I went about a tenth away from the end of the inch.

Interviewer: So, how did you decide to...to stop at sort of this...after this first interval (pointed to the point on the ruler just before the zero point that Kyle had previously indicated) why didn't you go up a second one?

Kyle: Um...well, from here to there (pointed to his first inch unit on the curve) it's hardly curving at all, so to straighten it out would just be like going...

Interviewer: a very minimal amount...

Kyle: Yeah.

Both Ned and Kyle applied compression intuition within the chord curved unit iteration strategy by mentally straightening parts of the curve for Tasks 9 and 10.

Attending to symmetry when measuring a curve with a ruler. As students measured the curves with a ruler for Tasks 9 and 10, some students' strategies suggested that they recognized symmetry in the shape of the curve (Figures 66 and 67). These students measured only parts of each shape, such as only the curved segments on the left-hand side of the "doorway" for Task 9 or half of the partial circle shaped curve for Task 10, rather than directly measuring the entire curve. There were nine instances in which students' strategies for measuring a curve with a ruler reflected an attention to symmetry: Ned (Grade 6, CRM Group), Rose (Grade 6, CRM Group), Trent (Grade 6, CRM Group), Rick (Grade 8, ICPM Group), Ruth (Grade 8, ICPM Group), Scott (Grade 10, ALM Group), and Kyle (Grade 10, ALM Group). The figures below illustrate Trent (Figure 67) and Rick's (Figure 68) attention to symmetry while measuring a curve with a ruler.

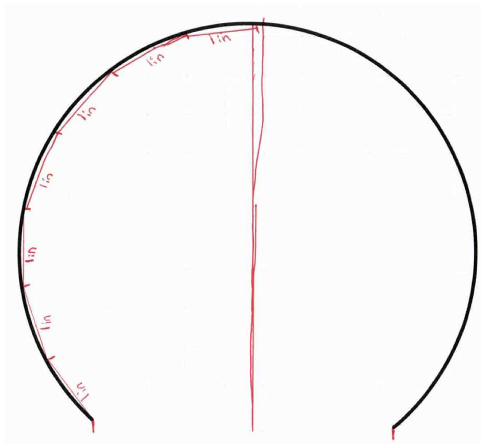


Figure 67. Trent's attention to symmetry while measuring a curve with a ruler.

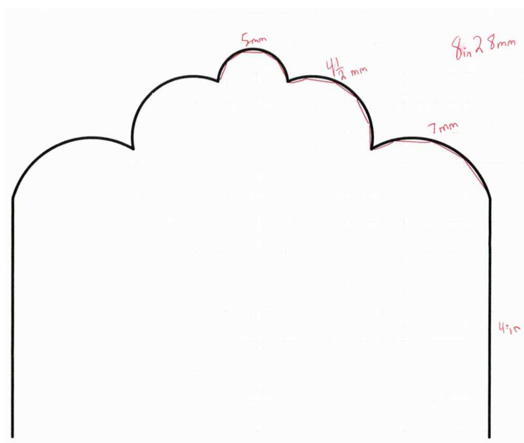


Figure 68. Rick's attention to symmetry while measuring a curve with a ruler.

Relating Analytical and Intuitive Thinking for Measuring Curves to the Length LT

Students' intuitions and analytical strategies for measuring a curve with a ruler were tracked within and across the length LT groups. In Figure 69, the darkest shade represents the LT group for which an intuition or analytical strategy occurred most often.

Tasks 9 & 10: Intuitions and Analytical Strategies	Chord Iteration Strategy	Tangent Iteration Strategy	Path Intersection Iteration Strategy	Adjusting Point of Tangency Iteration Strategy	Modified Circumference Formula	Tangent Curved Unit Iteration Strategy	Chord Curved Unit Iteration Strategy	Compression Intuition	Attended to Symmetry
CLM Group	3	4	0	2	0	0	0	0	0
CRM Group	2	2	1	2	0	0	2	2	4
ICPM Group	2	0	0	1	2	2	0	2	3
ALM Group	2	0	0	2	2	0	2	2	2
Totals	9	6	1	7	4	2	4	6	9

Figure 69. Patterns of intuition and analytical strategy use for measuring curves within and across LT groups.

Figure 69 shows developmental patterns for analytical strategies, intuition, and attention to symmetry across LT groups. I describe these patterns below, beginning with Table 20.

Table 20

Relating Intuitive and Analytical Strategies for Measuring a Curve with a Ruler (Tasks 9 and 10) to the Length LT Level Groups

	CLM Group	CRM Group	ICPM Group	ALM Group	Totals
Chord iteration	33.33%* (3)	22.22% (2)	22.22% (2)	22.22% (2)	9
Tangent iteration	66.67% (4)	33.33% (2)	0.00% (0)	0.00% (0)	6
Path intersection	0.00% (0)	100.00% (1)	0.00% (0)	0.00% (0)	1
Adjusting tangency	28.57% (2)	28.57% (2)	14.29% (1)	28.57% (2)	7
Modified circumference	0.00% (0)	0.00% (0)	50.00% (2)	50.00% (2)	4
Tangent curved unit	0.00% (0)	0.00% (0)	100.00% (2)	0.00% (0)	2
Chord curved unit	0.00% (0)	50.00% (2)	0.00% (0)	50.00% (2)	4
Compression intuition	0.00% (0)	33.33% (2)	33.33% (2)	33.33% (2)	6
Attended to symmetry	0.00% (0)	44.44% (4)	33.33% (3)	22.22 (2)	9

* 33.33% of the instances in which the chord iteration strategy was used occurred when CLM students were measuring a curve with a ruler (Tasks 9 and 10).

Patterns in Strategies for Measuring a Curve with a Ruler within LT Groups

Within the CLM level group, only the tangent, chord, and adjusting point of tangency analytical strategies were observed for measuring a curve with a ruler. At the CRM level, the chord, tangent, path intersection, adjusting point of tangency, and chord curved unit iteration strategies were observed. Also at this level, the use of the compression intuition appeared and instances of attention to symmetry occurred. Within the ICPM and ALM levels, participants used the chord iteration and adjusting point of tangency strategies. At these levels, participants also made use of the modified circumference formula, the compression intuition, and attention to symmetry as they measured the curves with a ruler. At the ICPM level, students also used the tangent curved unit iteration strategy, and ALM level students exhibited instances of the chord curved unit iteration strategy.

Patterns in Strategies for Measuring a Curve with a Ruler across LT Groups

The chord iteration and adjusting point of tangency iteration strategies were approximately evenly distributed across the length LT level groups. Use of the compression intuition, which occurred within the application of the chord curved unit and tangent curved unit iteration strategies, was evenly distributed across the CRM, ICPM, and ALM groups. The tangent iteration strategy was observed in the groups representing the lowest two LT levels included in the study, the CLM and CRM groups. Most of the instances of this strategy occurred in the CLM group, the lowest level group. I observed the application of the modified circumference formula strategy in the two groups representing the highest two LT levels, the ICPM and ALM groups. Instances in which students' responses reflected an attention to the symmetry of the shape of the curve occurred only within the groups representing the CRM, ICPM, and ALM levels.

I observed few instances of the analytical strategies participants used when measuring curves with standard units, a ruler (Tasks 9 and 10). Therefore, I collapsed codes into the same three thematic categories: direct measurement with no intuition use, direct measurement with embedded intuition use, and indirect measurement (see Table 11). Direct measurement includes the chord, tangent, path intersection, and adjusting point of tangency iteration strategies. Direct measurement with embedded intuition includes the tangent and chord unit iteration strategies. Indirect measurement includes both the modified circumference formula and the attention to symmetry codes. Table 21 shows the interaction between LT groups and these thematic categories,

Table 21

Interactions of Thematic Categories for Analytical Strategies for Measuring a Curve with Standard Units (Tasks 9 and 10)

	CLM Group	CRM Group	ICPM Group	ALM Group	Totals
Direct measurement (analytical strategies without embedded intuition)	39.13%* (9)	30.43% (7)	13.43% (3)	17.39% (4)	23
Direct measurement (analytical strategies with embedded intuition)	0.00% (0)	33.33% (2)	33.33% (2)	33.33% (2)	6
Indirect measurement	0.00% (0)	30.77% (4)	38.46% (5)	30.77% (4)	13

* 39.13% of the instances in which a direct measurement strategy without an intuition was used to measure a curve with a standard ruler occurred in the CLM level group.

Patterns of Analytical Strategies for Measuring Curves within LT Groups

CLM level participants relied only on direct measurement strategies without the use of embedded intuitions. Within the CRM level group, participants most frequently relied on direct measurement strategies without embedded intuitions; however, the use of direct measurement strategies with an embedded intuition and indirect measurement strategies were also observed. At the ICPM level, participants most often used indirect measurement strategies, but also occasionally used direct measurement strategies with or without an embedded intuition. The ALM level group most often exhibited indirect measurement strategies and direct measurement strategies without an embedded intuition. In this group, some instances of the use of direct measurement strategies with an embedded intuition were also observed.

Patterns of Analytical Strategies for Measuring Curves across LT Groups

The highest number of instances of the use of direct measurement strategies without an embedded intuition occurred within the CLM level group. The appearance of these strategies generally decreased in frequency across the LT levels as the levels increased in sophistication, but remained approximately the same at the ICPM and ALM levels. The use of direct measurement strategies with an embedded intuition and indirect measurement strategies occurred only within the CRM, ICPM, and ALM level groups, and the frequency of the appearance of these strategies was approximately evenly distributed across these three level groups.

Summary

In the sections below I summarize how students within each of the four LT level groups made use of intuitions and analytical strategies for rectilinear and curvilinear paths. This is followed by a section in which I describe and differentiate students' intuitive and analytical thinking for rectilinear and curvilinear paths across the LT level groups included in the study.

CLM Level Group

Participants in the CLM level group showed evidence of relying exclusively on intuitive statements to justify their comparisons of sets of rectilinear or curvilinear paths by length (Tasks 1, 2, 6A, 7, and 8A). When comparing rectilinear paths, CLM level participants most frequently exhibited the complexity intuition by talking about ordering the paths by the number of turns or segments. However, when comparing sets of curvilinear paths, students in the CLM group most frequently evoked the straightness intuition. Within the CLM level group, participants showed evidence of using intuitions in combination, intuitions in conflict, rejecting an intuition, and using rejected intuitions when ordering rectilinear and curvilinear paths by their lengths.

When measuring curves with a nonstandard unit, CLM level participants relied only on direct measurement strategies (Tasks 3, 4, 5, 6B, and 8B). Students in the CLM level group most often used the entire nonstandard unit, a 4-inch stick, as the unit when measuring lengths of curves. In addition, participants in this group showed evidence of using units and subunits, by fracturing a nonstandard unit once to fit a partial unit at the end of the curve. Students in the CLM level group most often (correctly) claimed that their comparison between a nonstandard unit and a curve resulted in an underestimate. CLM level students most often justified their claims about over- and underestimates when comparing a curve to a nonstandard unit using either an intuition or discussing their way of comparing the nonstandard unit and the curve. When measuring curves using standard units, a ruler, CLM level students used three direct measurement strategies they had used when comparing a nonstandard unit and a curve: the chord iteration, tangent iteration, and adjusting point of tangency iteration strategies.

CRM Level Group

The CRM level group students relied on intuitive statements as well as analytical strategies when comparing sets of rectilinear or curvilinear paths by their lengths and justifying those orderings (Tasks 1, 2, 6A, 7, and 8A). Most often, they ordered rectilinear paths using the complexity intuition, by attending to the number of turns or segments, and made judgments about the order of sets of curvilinear paths by their lengths by mentally transforming the paths into the same shape. CRM level participants used intuitions in combination; however, none of the participants at this level used intuitions in conflict, rejected an intuition, or used a rejected intuition.

When measuring curves with a nonstandard unit, CRM level participants relied on a direct measurement strategy or a direct measurement strategy with an embedded intuition (Tasks 3, 4, 5, 6B, and 8B). For these tasks, students in this group also showed evidence of applying mental units, mentally transforming units or segments of a curve, compensating for curvature by rounding up or down to account for an over- or underestimate, and fracturing units to make use of units and subunits for the purpose of increasing precision. However, they did not yet consistently show evidence of coordinating linear extent with other attributes, such as curve, by using smaller units to increase precision around a tighter curve.

CRM level participants claimed to have over- or underestimated approximately an equal number of times when reflecting on their ways of comparing a nonstandard unit to a curve. These responses were evenly split between correct and incorrect acknowledgements of over- or underestimates. When justifying why they thought they had over- or underestimated, participants at the CRM level most often discussed the analytical strategy they had used for operating on the nonstandard unit. Within the CRM level group, when measuring curves with standard units, using a ruler, participants used four direct measurement strategies they had used when comparing a curve and a nonstandard unit: the chord iteration, tangent iteration, path intersection, and adjusting point of tangency iteration strategies. Also at this level, when measuring a curve with a ruler, students exhibited the use of the compression intuition and a direct measurement strategy with an embedded intuition, the chord curved unit iteration strategy. Participants in the CRM level group also used strategies to measure curves with rulers that reflected attention to symmetry.

ICPM Level Group

Students in the ICPM level group relied on intuitive statements as well as analytical strategies when comparing sets of rectilinear or curvilinear paths by their lengths and justifying those orderings (Tasks 1, 2, 6A, 7, and 8A). They most frequently evoked the straightness intuition when defending their orderings of sets of rectilinear or curvilinear paths by their lengths. When comparing sets of rectilinear paths (Tasks 1 and 2), IPCM level students showed evidence of using intuitions in combination. However, when comparing sets of curvilinear paths (Tasks 6A, 7, and 8A), they showed evidence of using intuitions in combination as well as experiencing conflicts in intuition use, rejecting an intuition, and using a rejected intuition.

Within the ICPM level group, students most often relied on direct measurement strategies when measuring curves with a nonstandard unit (Tasks 3, 4, 5, 6B, and 8B). However, for these tasks they also showed evidence of using direct measurement strategies with an embedded intuition and an indirect measurement strategy, applying a modified version of the formula for the circumference of a circle. For the tasks involving curves, the ICPM level participants often mentally transformed the nonstandard unit or segments of the curve, fractured the nonstandard unit to make use of units and subunits, and showed evidence of coordinating linear extent with other attributes, such as curve, by using smaller units around a tight curve. Within the ICPM level group, students claimed to have over- or underestimated when comparing a nonstandard unit to a curve approximately the same number of times. Most of these claims were either correct or could not be determined to be either correct or incorrect, and ICPM level participants most often defended why they thought they had over- or underestimated using an intuition. When measuring a curve with a ruler, ICPM level participants exhibited two direct measurement strategies that were observed as students compared a curve and a nonstandard unit: the chord iteration and adjusting point of tangency iteration strategies. They also exhibited the use of the compression intuition and an analytical strategy with an embedded intuition, the tangent curved unit iteration strategy. In addition, ICPM level participants exhibited the use of an indirect measurement strategy, applying a modified circumference formula, and attention to symmetry when measuring a curve with a ruler.

ALM Level Group

Participants at the ALM level, the group representing the highest level of the length LT, relied on intuitions and analytical strategies when comparing sets of rectilinear or curvilinear paths by their lengths and defending those orderings (Tasks 1, 2, 6A, 7, and 8A). The ALM level participants most often evoked the compression intuition, by discussing mentally straightening paths that were bent or bending paths that were straight, or the straightness intuition when comparing rectilinear or curvilinear paths by their lengths. At this level, students showed evidence of using intuitions in combination when comparing rectilinear or curvilinear paths. However, they showed evidence of experiencing conflicts among intuitions, rejecting an intuition, and using a rejecting intuition only when comparing rectilinear paths (Tasks 1 and 2).

When measuring curves with a nonstandard unit (Tasks 3, 4, 5, 6B, and 8B), students at the ALM level relied most often on direct measurement strategies, but they also showed evidence of using direct measurement strategies with an embedded intuition and applying an indirect measurement strategy by using a modified circumference formula. Students at the ALM level also compensated for curvature by rounding a measurement up or down to account for an over- or underestimate. In addition, ALM level students also mentally transformed the nonstandard unit or segments of the curve and showed evidence of coordinating linear extent with another attribute, curvature, by fracturing nonstandard units around tight curves to increase precision. When comparing a curve to a nonstandard unit, participants in the ALM level group claimed to have underestimated or claimed to have neither over- nor underestimated approximately the same number of times. These claims were most frequently either correct or could not be determined to be correct or incorrect. ALM level participants most often justified why they thought they had over- or underestimated by discussing the analytical strategy they had used for comparing the straight object, the nonstandard unit, and the curve.

Within the ALM level group when measuring a curve with a ruler, participants used two direct measurement strategies when comparing a curve and a straight object: the chord iteration and adjusting point of tangency iteration strategies. ALM level participants also exhibited the compression intuition and the application of a direct measurement strategy with an embedded intuition: the chord curved unit iteration strategy when measuring curves with a ruler. In addition, when measuring curves with a ruler, students at the ALM level applied an indirect measurement strategy, using a modified circumference formula, and attended to symmetry.

Summary of Intuitive and Analytical Thinking across LT Level Groups

Parallel to prior research, students used four main types of intuitions when comparing rectilinear paths by their lengths: straightness, complexity, detour, and compression (Chiu, 1996). When comparing curvilinear paths by the lengths, the participants of this study exhibited these same four main types of intuitions as well as a fifth intuition, the curve tightness intuition. Across all four length LT level groups, students most often evoked the complexity intuition, by attending to the number of segments or turns in the paths, when ordering rectilinear paths by their lengths (Tasks 1 and 2). However, the straightness intuition and compression intuition, which involved mentally bending paths that were straight or straightening paths that were bent, were the most frequently used intuitions when comparing curvilinear paths by their lengths (Tasks 6A, 7, and 8A).

The four participants at the CLM level, the group that represented the lowest LT level included in the present study, exhibited the highest frequency of intuition use when comparing rectilinear paths by their lengths (Tasks 1 and 2). However, the ICPM level group exhibited the highest frequency of intuition use when comparing curvilinear paths by their lengths (Tasks 6A, 7, and 8A). Students at the CLM, ICPM, and ALM levels exhibited conflicting intuitions, the rejection of an intuition, and the use of a rejected intuition when comparing rectilinear or curvilinear paths by their lengths (Tasks 1, 2, 6A, 7, and 8A). The use of intuitions in combination for comparing sets of rectilinear or curvilinear paths was observed across all four length LT level groups. When comparing rectilinear or curvilinear paths by their lengths, only the CLM level group relied solely on the use of intuitions. Students at the ALM level increasingly relied on mentally transforming rectilinear or curvilinear paths into the same shape for the purpose of comparing by lengths and were less likely to order rectilinear or curvilinear paths according to the number of segments or turns than students at the CLM, CRM, and ICPM levels.

When measuring curves with a nonstandard unit, a 4-inch stick (Tasks 3, 4, 5, 6B, and 8B), students in the lowest LT level group, the CLM group, used only direct measurement strategies. By the next level of the length LT, the CRM level, students used direct measurement strategies as well as direct measurement strategies with embedded intuitions. At the ICPM and ALM levels, students used direct measurement strategies with and without embedded intuitions, as well as an indirect measurement strategy, applying a modified circumference formula to make a claim about curve length. The use of direct measurement strategies (without embedded intuitions) was at a peak in the CLM level group, and decreased in frequency of appearance within each group as the levels increased in sophistication.

Participants in the lowest LT level group, the CLM group, exhibited the highest number of instances of using the whole stick as the unit when measuring a curve with a nonstandard unit (Tasks 3, 4, 5, 6A, and 8A). This strategy decreased across the LT level groups as the levels increased in sophistication. Students in the CRM level group applied mental units more often than any other LT level group on the set of 10 interview tasks. The fewest instances of fracturing units occurred within the CLM level and increased across the LT level groups as the levels became increasingly sophisticated. Instances of fracturing units in general increased from the CLM to the CRM level. More specifically, the occurrences of fracturing units in the tightest part of the curve and fracturing units along the entire curve increased from the CRM to the ICPM level, and remained approximately constant from the ICPM to the ALM levels.

When reflecting on their comparison between a nonstandard unit and a curve (Tasks 3, 4, 5, 6A, and 8A), students within all LT level groups most often claimed their comparison resulted in an underestimate. CLM level students most often correctly noted that their comparison resulted in an over- or underestimate. The frequency of the appearance of acknowledgements of a comparison as an overestimate was highest at the ICPM level, and the instances of claims that a comparison did not results in an over- or underestimate was highest at the ALM level. The frequency of incorrect claims that a comparison resulted in an over- or underestimate was at a maximum for the CRM level group, and claims that an answer was neither correct nor incorrect was highest for the ICPM level group. Across the four length LT level groups, clear developmental patterns were not observed in participants' justifications of their claims about why they thought they had over- or underestimated.

When measuring curves with standard units, a ruler (Tasks 9 and 10), I observed the direct measurement strategies of chord iteration and adjusting point of tangency most often across the four length LT level groups. Attention to symmetry, the use of the compression intuition and indirect measurement strategies with embedded intuitions, the tangent and chord curved unit iteration strategies, were evenly distributed across the CRM, ICPM, and ALM levels.

CHAPTER V

CONCLUSIONS AND IMPLICATIONS

Overview

In this study I explored elementary, middle, and secondary students' intuitive and analytical thinking for rectilinear and curvilinear paths. By examining intuitive and analytical thinking as developmental phenomena, and in tandem with concept growth along a hypothetical learning trajectory (LT) for length measurement (Clements et al., in press), this study contributed to ongoing conversations in multiple disciplines: mathematics education, science education, and psychology. In this chapter, I will first compare results involving length measurement, derived from the written length LT-based assessment administered to 82 participants, with prior research in mathematics education as well as recommendations for the teaching and learning of measurement from researchers in science education. Next, I will discuss how findings speak to psychological foundations of path length intuitions, and the development of those intuitions across the elementary, middle, and secondary years. I will then discuss how these findings compare to hypothesized concepts and processes outlined in the four length LT levels included in the study: Consistent Length Measurer (CLM), Conceptual Ruler Measurer (CRM), Integrated Conceptual Path Measurer (ICPM), and Abstract Length Measurer (ALM). Finally, I will discuss limitations and examine implications for teaching and research.

Findings Related to Length Measurement

Comparing with the National Assessment of Educational Progress (NAEP)

The results related to length measurement are consistent with results from the 2000 and 1996 NAEP. These findings related to length measurement were derived from the 7-task written length LT-based assessment, which was administered to a total of 82 participants from Grades 4, 6, 8, and 10. In the present study, when shown an image of a paper strip placed along a broken section of a ruler and asked to determine the length of the paper strip (Tasks 1 and 2), 23% and 32% of the Grade 4 students answered correctly on Tasks 1 and 2, respectively. This is similar to the performance of Grade 4 students reported for both the 2000 and 1996 NAEP, with 25% and 22% answering correctly in 2000 and 1996, respectively (Kloosterman et al., 2004; Sowder et al., 2004). The Grade 8 participants' performance on the broken ruler tasks, with 75% answering correctly on both Tasks 1 and 2, was better than the performance of Grade 8 students reported for both the 2000 and 1996 NAEP, with 40% and 63%, respectively. Participants from Grade 10 in the present study exhibited similar performance on the broken ruler tasks, with 72% and 89% answering correctly for Tasks 1 and 2, respectively to the Grade 12 students from the 1996 NAEP, with 83% answering correctly. Furthermore, these findings support the long-standing record established by NAEP, which shows that students at the elementary, middle, and secondary levels do not connect numerical measurement with the process of unit iteration (Barrett & Clements, 2003; Battista, 2006; Clements, Battista, Sarama, Swaminathan, McMillen, 1997). That is, they do not understand that a ruler represents a collection of iterated units.

Comparing with Prior Research on an LT for Length Measurement

The findings concerning length measurement reported here address a significant gap in the literature with respect to the length LT levels exhibited by a cross-section of elementary, middle, and secondary level students. Prior to this study, elementary children's thinking and learning for length measurement, as measured by the LT for length measurement was described (Clements et al., in press). According to Clements et al., (in press), when exposed to specific instruction designed to support students' concept growth along the length LT, students in Kindergarten predominantly exhibited direct and indirect comparison strategies (LDC and ILC levels) and strategies for measuring by spanning an object with length units laid end-to-end without gaps or overlaps (EE level). By Grade 1 and early on in Grade 2, students most often exhibited strategies for measuring by laying length units end-to-end to span an object (EE level) or by repeating or iterating a length unit (LURR level). By the end of Grade 2 and early on in Grade 3, students predominantly relied on unit iteration (LURR level) and increasingly exhibited an ability to measure straight paths consistently, use equal-length units, understand the zero point on a ruler, and partition units (CLM level). Late in Grade 3 and into Grade 4, students also began to demonstrate some instances of applying an "internal" measurement tool by mentally iterating internal units of length or partitioning a length into equal-length parts and projecting or translating given lengths to determine missing lengths (CRM level). Also Grade 4 students exhibited some instances of integrating and comparing sets of units along each section of a bent path and constructing smaller units for the purpose of increasing precision (ICPM level). In addition, Grade 4 students exhibited some instances of operating internally collections of complex paths and exhibiting a continuous sense of space (ALM level). This same trend was also seen by Clements et al. in Grade 5 participants.

The Grade 4 students in the present study exhibited length LT levels in ways similar to results reported by Clements et al. (in press). That is, the Grade 4 participants mainly exhibited strategies for measuring based on unit iteration (LURR level) or an understanding of the zero point on a ruler (CLM level) when resolving broken ruler tasks (with 50% and 64% using LURR strategies on Task 1 and 2, respectively, and 23% and 32% using CLM level strategies on Tasks 1 and 2, respectively). Grade 4 students showed some evidence of translating given lengths to determine missing lengths (CRM level, with 9% on Task 3) and integrating and comparing sets of units along each section of a bent path and constructing smaller units to increase precision (ICPM level, with 23% each on Tasks 5 and 6).

The present study extends the work of Clements et al. (in press) by describing the concepts and processes, which define particular levels of the LT for length measurement, that students use beyond the elementary years into middle and secondary school. Results reported here indicate that most of the Grade 6 students in the present study exhibited strategies for measuring based on the iteration of length units (LURR level) or an understanding of the zero point on a ruler (CLM level) in the contexts in which those concepts and processes (or the levels) were relevant (with 27% and 41%, respectively on Task 1 and 32% and 45%, respectively on Task 2). The Grade 6 participants translated given lengths to determine missing lengths (CRM level) more often than Grade 4 participants (with 27% and 9% on Tasks 3 and 4, respectively). Furthermore, the participants from Grade 6 exhibited more instances of integrating and comparing sets of units along each section of a bent path and constructing smaller units to increase precision (ICPM level, with 9% and 50% for Tasks 5 and 6, respectively) than the Grade 4 participants.

The Grade 8 students most often exhibited strategies that demonstrated an understanding of the zero point on a ruler to resolve broken ruler tasks (CLM level, with 75% each on Tasks 1 and 2). Participants in Grade 8 also translated given lengths to determine missing lengths (CRM level, with 75% and 50% on Tasks 3 and 4, respectively) more often than the participants in Grade 6. Furthermore, the Grade 8 students increasingly integrated and compared sets of units along each section of a bent path and constructed smaller units to increase precision (ICPM level, with 45% and 65% for Tasks 5 and 6, respectively), and showed some evidence of operating internally on collections of units of units as well as collections of complex paths and exhibiting a continuous sense of space (ALM level, with 10% each for Tasks 5 and 6).

Grade 10 students exhibited a pattern similar to the Grade 8 level participants for relying on strategies that demonstrated an understanding of the zero point on a ruler to resolve broken ruler tasks (CLM level, with 72% and 89% for Tasks 1 and 2, respectively). The Grade 10 participants increasingly translated given lengths to determine missing lengths (CRM level, with 100% and 72% for Tasks 3 and 4, respectively) and operated internally on collections of units of units as well as collections of complex paths and exhibited a continuous sense of space (ALM level, with 33% and 44% for Tasks 5 and 6, respectively).

Researchers previously reported observing LURR and CLM level thinking predominantly in Grades 2 and 3 (Clements et al., in press); however, the results of the present study suggest that these levels are also relevant for students beyond elementary school and into Grades 6, 8, and 10. Taken together, these findings suggest that students continue to progress through the levels of the length LT beyond their elementary years into middle and secondary school in a typical educational context in the Midwestern United States.

Comparing with a Learning Progression in Science Education

Results reported here concerning length measurement, which were derived from the written length LT-based assessment, are in contrast to recommendations for the teaching and learning of measurement articulated in a learning progression for the atomic-molecular theory of matter (LP for AMTM) in science education (National Research Council [NRC], 2007; see also Smith, Wiser, Anderson, and Krajcik, 2006). In the LP for AMTM, it is recommended that, in Kindergarten through Grade 2, children should learn that “good measurements use iterations of a fixed unit (including fractional parts of that unit) to cover the measured space completely (no gaps)” (NRC, 2007, p. 364). This recommendation spans the LURR and CLM levels of the length LT (see Table 1 in Chapter 2). Findings from Tasks 1 and 2 of the written length LT-based assessment in the present study suggest that, at Grade 4 over half of students (50% and 64% for Tasks 1 and 2, respectively) show evidence of measuring by repeating or iterating a unit, which is evidence of LURR level thinking. However few Grade 4 students (23% and 32% for Tasks 1 and 2, respectively) show evidence of possessing well-developed ideas about unit iteration in terms of understanding the zero point on the ruler and seeing a ruler as a collection of iterated units, which is consistent with the CLM level. Therefore, these findings suggest that Smith, Wiser, Anderson, and Krajcik’s (2006) recommendation for Kindergarten through Grade 2 measurement is not currently being met in a typical educational context in the Midwestern United States.

Addressing Research Question 1

The first research question that framed this study addressed the intuitions and analytical strategies that students use when thinking about rectilinear or curvilinear paths: What intuitions and analytical strategies do students use when comparing sets of rectilinear or curvilinear paths by length? The Grade 4, 6, 8, and 10 participants exhibited four main intuitions for comparing rectilinear paths by length that were identified in prior research with Grade 6 students: complexity, compression, detour, and straightness (Chiu, 1996). This suggests that these intuitions for path length may be extensive beyond the scope indicated by prior research. Findings from the present study extend the body of literature on path length intuition by revealing that students operate on five main intuitions for comparing curvilinear paths by length, which include the four main intuitions established in the literature and one new intuition: the curve tightness intuition.

In the present study, students who used the curve tightness intuition discussed a particular curve as being longer than another because it was curved in more or had more curve. Students who exhibited the curve tightness intuition did not (a) attend to the straightness of a path, the straightness intuition; (b) discuss a process of straightening or bending a curve, the compression intuition; (c) discuss a path as deviating away from the destination more than another, the detour intuition; or (d) attend to the number of turns or segments in the paths, the complexity intuition. That is, the curve tightness intuition was exhibited by students' responses that reflected an attention to the quality of a path as being curved and did not fit within any of the other four categories of path length intuition.

The curve tightness intuition may be psychologically grounded in intuitions discussed in prior research, such as the straightness or compression intuitions (Chiu, 1996). For example, students who intuitively know that the shortest path is a straight line may also intuitively know that a path with a slight curve is shorter than a path with a tight curve. Alternatively, students who intuitively know that a coil or string that has been compressed may also intuitively know that a tight curve is more compressed than a curve that is wide and conclude that the tight curve is longer than the wide curve even though the distance between the endpoints of the wide curve is greater than the tight curve. However, despite the potentially common psychological foundations with other intuitions, the curve tightness intuition appeared as a qualitatively different category of responses within the larger thematic category of intuitive thinking (Fischbein, 1987).

Furthermore, the results reported here reveal that intuitions that were present in Grade 6 students' thinking (Chiu, 1996) were also present in Grade 4 students' thinking and persist beyond Grade 6 into Grades 8 and 10. Overall, participants most often evoked the complexity intuition by attending to the number of turns or segments in a particular path when justifying their claims about the order of rectilinear paths by length. This finding is consistent with other studies in psychology that have shown that the complexity intuition is robust across a wide age range and across a wide variety of contexts (e.g., Barrett & Clements, 2003; Kosslyn, Pick, & Fariello, 1974; Luria, Kinney, & Weissman, 1967; Pressey, 1974; Thorndyke, 1981).

Although the tasks involving comparing curvilinear paths were designed to be parallel to the rectilinear path length comparison tasks, participants most often evoked the straightness intuition or compression intuition with curves, which involves mentally bending paths that are straight or mentally straightening paths that are bent. This suggests that, although children in Grades 4, 6, 8, and 10 possess some of the same intuitions for rectilinear and curvilinear paths, the presence of curve introduced intuitive interference that is not present when the paths are rectilinear.

Results from the present study also suggest that, when measuring curves with standard or nonstandard units, students exhibit analytical thinking by applying indirect measurement strategies, such as using a modified circumference formula or attending to symmetry, or direct measurement strategies that may or may not have an embedded intuition. Furthermore, students exhibit strategies for operating on units when measuring curves, which provides evidence about how they are able to coordinate linear extent with other attributes, such as curve (Clements et al., in press). In the present study, students who did not coordinate linear extent with curve exhibited strategies of not fracturing units at all or fracturing a unit once for the purpose of increasing precision. However, students who had developed the ability to coordinate linear extent with curve exhibited instances of fracturing the nonstandard unit in the tightest part of the curve or along the entire curve.

Addressing Research Question 2

The second research question that guided the design, data collection, and data analysis in this study concerns how students' use of intuitive and analytical thinking for rectilinear and curvilinear path develops across Grades 4 through 10: How does students' use of intuitive and analytical thinking for path length change or develop across levels of sophistication for length measurement? The sections below describe conclusions regarding the developmental patterns observed across a subset of levels of the LT for length measurement included in the present study, which are, in order of increasing sophistication, the CLM, CRM, ICPM, and ALM levels.

Results indicate that participants at all four length LT levels exhibited the four main intuitions for rectilinear paths (Chiu, 1996) and five main intuitions for curvilinear paths, which includes a new intuition, the curve tightness intuition. However, different length LT level groups exhibited different patterns of intuition use. Specifically, the CLM group, the lowest length LT level group included in the present study, relied only on intuitive statements when comparing rectilinear or curvilinear paths by their lengths. Students at the CRM, ICPM, and ALM levels all used intuitions as well as analytical strategies when comparing sets of rectilinear or curvilinear paths. Participants at the CLM level most often relied on the complexity intuition, ordering rectilinear paths by the number of segments or turns, and the appearance of this intuition decreased across LT level groups as the levels increased in sophistication. In contrast, students at the ALM level increasingly mentally transformed rectilinear or curvilinear paths into the same shape, which shows evidence of evoking the compression intuition, for the purpose of comparing the paths by length.

When comparing a curve to a nonstandard unit, CLM level participants relied exclusively on direct measurement strategies without embedded intuitions, exhibited the highest number of instances of using the whole stick as a unit, and showed the fewest instances of fracturing the nonstandard unit to increase precision. Both strategies of directly measuring without an embedded intuition and using the whole stick decreased across the LT level groups as the levels increased in sophistication. By the CRM level, students applied direct measurement strategies with or without embedded intuitions and applied mental units more often than any other LT level group. In addition, CRM level students increasingly fractured units for the purpose of increasing precision when a full stick did not fit in the position of the final stick unit. At the ICPM and ALM levels, students used direct measurement strategies with or without embedded intuitions as well as indirect measurement strategies. Also at the ICPM and ALM levels, participants increasingly fractured units, especially around the tightest parts of the curve or around the entire curve, showing evidence of coordinating linear extent with other features, which in this case was curvature.

The results reported here suggest that the tasks included in this study effectively differentiated students' thinking at different levels of the length LT. Furthermore, these findings are consistent with Fischbein's theory of intuition (1987), in which he described intuition as a developmental phenomenon. Participants who exhibited different levels of sophistication, as measured by the LT for length measurement, also exhibited different ways of evoking intuitions in terms of the use of (a) intuitions and analytical strategies overall, (b) each individual intuition, and (c) analytical strategies with embedded intuitions.

Furthermore, the results reported here confirm some of the conjectured concepts and processes outlined at different levels of the LT (Clements et al., in press). For example, it was conjectured that CLM level students possessed integrated counting and iterating schemes that allow for the concurrent iteration of a unit and subdivision of the unit. This was confirmed as CLM level students typically exhibited instances of operating with a combination of units and parts of units when measuring a curve with a nonstandard unit. At the CRM level of the LT for length measurement, it was hypothesized that students mentally partition lengths by projecting a mental unit, a ruler, or a sequence of units onto an unpartitioned object. This was supported by the results of the present study; the highest frequency of the appearance of the application of mental units occurred within the CRM level group. At the ICPM level of the length LT, it was conjectured that students would coordinate other measures with linear measures, such as angle, curvature, or time. In the present study, the ICPM level group exhibited increased instances of fracturing the unit in the tightest part of the curve and fracturing the unit along the entire curve, showing evidence of coordinating linear measures with curvature. Finally, at the ALM level of the length LT, it was hypothesized that students had developed a continuous sense of length. This was confirmed by the results of the present study as the participants within the ALM level group increasingly relied on mentally transforming rectilinear or curvilinear paths into the same shape for the purpose of comparing by length.

Table 22 summarizes extensions to the existing LT for length measurement (Clements et al., in press) with respect to intuitive and analytical thinking for rectilinear and curvilinear path length. In the following table, these extensions to the LT are italicized.

Table 22

Extending Intuitive and Analytical Thinking for Path Length to an LT for Length Measurement

Developmental Progression	Mental Actions on Objects
Consistent Length Measurer (CLM) Measures straight paths consistently, uses equal-length units, understands the zero point on the ruler, and can partition units to make use of units and subunits May not be perturbed by geometric inconsistencies - *Most often orders collections of rectilinear paths by the number of turns or segments in the path* - *Relies on direct measurement strategies without making use of intuition when measuring curves with nonstandard units*	Integrates intervals and tick marks indicating endpoints of intervals to establish linear quantity Integrated counting and iterating schemes allow for the concurrent iteration of a unit and subdivision of a unit - *Relies exclusively on intuitive statements to justify orderings of sets of rectilinear or curvilinear paths by their lengths* - *May fracture a unit to make use of units and subunits for the purpose of increasing precision, but does not yet coordinate linear extent with other attributes, such as curve*
Conceptual Ruler Measurer (CRM) Has an "internal" measurement tool; mentally iterates internal units of length or partitions a length into equal-length parts Projects or translates given lengths to determine missing lengths Notices geometric inconsistencies - *Most often orders collections of rectilinear paths by the number of turns or segments* - *Occasionally relies on direct measurement strategies without making use of intuition when measuring curves* - *May rely on analytical strategies with embedded intuition by mentally transforming units or segments of a*	Mentally partitions lengths by projecting a mental unit, a ruler, or a sequence of units onto an unpartitioned object Increasingly uses multiplicative reasoning when comparing - *Relies on intuitive statements as well as analytical strategies when comparing sets of rectilinear or curvilinear paths by their lengths and justifying those orderings* - *Makes judgments about the order of sets of curvilinear paths by their lengths by mentally transforming the paths into the same shape* - *Fractures a unit to make use of*

curve - *May compensate for curvature by rounding an approximation for the length of a curve up or down for an over- or underestimate* - *Attends to symmetry in path shape* - *Does not consistently correctly acknowledge an over- or underestimate when approximating the length of a curve*	*units and subunits, and may begin to show evidence of coordinating linear extent with other attributes, such as curve, by occasionally using smaller units to increase precision around a tighter curve* - *Applies mental units when comparing two or more rectilinear paths or curves by lengths*
Integrated Conceptual Path Measurer (ICPM) In the context of a fixed perimeter or fixed path length task, children at the ICPM level are able to compensate for changes made to one side of a figure by adjusting other sides to maintain the fixed overall length. Shows well-developed ideas about precision, such as constructing smaller units to increase precision - *Relies on direct measurement strategies without making use of intuition when measuring* - *When measuring curves, relies on analytical strategies with embedded intuition by mentally transforming units or segments of a curve* - *When measuring curves, relies on indirect measurement strategies, such as applying a formula or attending to symmetry*	Integrates and compares sets of units along each section of a bent path; Regards a group of units as a flexible object, a "string" of units wrapped around the entire perimeter or along the entire path Copes sub- and superordinate units Coordinates other measures with linear measures, such as angle, curve, or time - *Relies on intuitions and analytical strategies when comparing sets of rectilinear or curvilinear paths and justifying those orderings* - *Fractures a unit to make use of units and subunits, coordinates linear extent with other attributes, such as curve, by using smaller units to increase precision around a tighter curve*
Abstract Length Measurer (ALM) Synthesizes sets of figures based on perimeter to formulate and justify a valid argument; Determines perimeter or path length, attending to divisions of units including non-integer values; explains the subdivision process is potentially unlimited - *May rely on direct measurement strategies without making use of intuition when measuring curves* - *When measuring curves, uses analytical strategies with embedded*	Develops a continuous sense of length Engages dynamic imagery to coordinate and operate internally on collections of units of units as well as collections of complex paths - *Relies on intuitive statements as well as analytical strategies when comparing sets of rectilinear or curvilinear paths by their lengths and justifying those orderings*

intuitions by mentally transforming units or segments of a curve - When measuring curves, relies on indirect measurement strategies, such as applying a formula or attending to symmetry	- Increasingly relies on mentally transforming rectilinear or curvilinear paths into the same shape for the purpose of comparing by length

Note: For the complete LT for length measurement, including anticipated misconceptions for each level, see Clements et al. (in press).

As summarized in Table 22 above, participants at all LT levels exhibited intuitions for path length; however, the application of analytical strategies and analytical strategies with embedded intuitions was not observed within all levels. These findings do not suggest that intuition is a less sophisticated cognition than analysis. Rather, Table 22 indicates that the application of exclusively intuitive *or* analytical thinking alone, observed mainly within the CLM and CRM levels, is less sophisticated than the application of intuitive *and* analytical thinking embedded within a single strategy, which was observed most often at the ICPM and ALM levels.

Furthermore, the findings summarized in Table 22 suggest that a hierarchy may exist for some of the specific intuitions for path length discussed here. In particular, the peak of the complexity intuition at the CLM level (the lowest LT level included in the study) and the pattern of decreasing frequency for the complexity intuition as the length levels increased in sophistication indicates that it is the least sophisticated intuition. On the other hand, the peak of the appearance of the compression intuition at the highest level included in the present study, the ALM level, indicates that it may be the most sophisticated intuition for path length. Clear developmental patterns were not observed for the detour and straightness intuitions; this suggests that, unlike the complexity and compression intuitions, these specific intuitions may not be hierarchical. A key implication of this finding is that there exists a developmental mechanism for describing connections between some intuitions for path length as well as connections between intuitive and analytical cognition. In the concluding sections of this chapter I discuss further implications of the hierarchical structure of path length intuition, which parallels the length LT, for both teaching and research.

In addition, close examination of analytical strategies for comparing paths, which were exhibited by students at the CRM, ICPM, and ALM levels, indicates that there may also exist a hierarchy of comparison strategies within the analytical thinking for path length. Specifically, direct and indirect comparison strategies were observed as some students superimposed pairs of paths to compare directly or compared indirectly using a finger span. Both of these strategies are consistent with the articulation of the observable behaviors that characterize the Length Comparer (LC) level of the length LT (Clements et al., in press), which is at least four levels below the predominant levels of the participants who exhibited them. According to the theory of Hierarchic Interactionalism (Clements & Sarama, 2007), LT levels build hierarchically out of previous levels and concepts and processes of lower levels are not abandoned. The CRM, ICPM, and ALM level participants' application of these LC-level comparison strategies indicates that some students fell back to using levels of thinking that were lower than their predominant LT level when resolving the path length comparison tasks; this task may have been novel to them, and this may have contributed to the tendency to drop back to a lower level of strategy.

The accumulating length comparison strategy, another analytical strategy observed in the present study, was exhibited by CRM, ICPM, and ALM level participants as they superimposed pairs of paths, and rotated one of the paths while accumulating the length of the first along the second. A belief that this is a valid strategy for comparing paths that are bent or curved requires not only conservation of length, but also mental actions and objects to integrate and compare sets of units along each section of a bent path. This use of the accumulating length comparison strategy is indicated in Table 22 above as fitting into the ICPM level of the length LT. This suggests that the accumulating length comparison strategy is a more sophisticated strategy than superimposing pairs of paths to compare directly or comparing indirectly using a finger span.

Limitations

Although the present study addressed critical outstanding questions about how patterns of students' intuitive and analytical thinking along with concept growth in one content domain, length measurement, it is not without limitations. One limitation of this study can be attributed to the inclusion of a written length LT-based assessment. By using a paper-pencil instrument I was able to assess a large sample of 82 students; however, my analysis was constrained to the observable strategies present in students' written responses to the items. At times, a student's response was unclear, and I was not able to make an inference about his or her level of sophistication for length measurement. Therefore, this instrument provided a limited opportunity to explore and substantiate claims about students' conceptions for length measurement.

A second limitation, which is a consequence of the design of this exploratory study, concerns the small number of students included in each grade and LT level. Because only four students were representative of each level, the conclusions about interactions among intuitive and analytical thinking for rectilinear and curvilinear paths with the levels of sophistication of a length LT that describes the growth of conceptual and procedural knowledge, indicate that developmental patterns may exist. However, a series of follow-up studies that focus on subsets of similar interview tasks included here, such as the rectilinear path length comparison tasks (Interview 1 Tasks 1 and 2), could validate the existence of these developmental patterns with larger and more diverse populations of students using statistical inference.

Other limitations of the study can be attributed to the methods of participant selection and the cross-sectional design used to examine students' intuitive and analytical thinking for rectilinear and curvilinear paths as a developmental phenomenon. Because participants at the CLM, CRM, ICPM, and ALM levels were sought from Grades 4, 6, 8, and 10, all but one of the participants performed in the top half of the class on the written length LT-based assessment. Therefore, the sample of 16 interview participants cannot be regarded as a representative sample with respect to the diversity in thinking present in typical Grade 4, 6, 8, and 10 classes in the Midwest. In addition, because I did not follow students longitudinally to document shifts in their use of intuitions and analytical strategies as they progressed through the levels of sophistication in the length LT, the findings reported here must be regarded as suggestive of development. These findings should be validated in a follow-up study that makes use of a longitudinal methodology.

Finally, inferences about participants' intuitive and analytical thinking for rectilinear and curvilinear path length were derived from their observable statements, gestures, and manipulations of tools during the structured, task-based interviews. The validity of these inferences is constrained by the quality of the tasks and the probing follow-up questions in the interview protocol. Efforts to ensure valid data included deriving tasks and pre-planned follow-up questions for the interview protocols from prior research (Clements et al., in press; Chiu, 1996; Grugnetti & Rizza, 2004) and refining those tasks and the interview protocols through pilot work.

Implications for Teaching

An assumption of this study and prior research on intuition (Chiu, 1996; Fischbein, 1987) is that people possess some intuitions that are mathematically productive and others that are not. Mathematically productive intuitions are those that can serve as a pedagogical starting point for the teaching of key mathematical concepts. For example, a combination of the detour and straightness intuitions, that a path that turns and goes out of the way is longer than a path that is straight, can serve as an intuitive foundation for the development of thinking about the triangle inequality (Chiu, 1996). The use of the complexity intuition, attending to the number of segments or turns, to rank rectilinear or curvilinear paths by length is an example of an application of an intuition that is not mathematically productive. In this study, the use of the complexity intuition appeared most often at the CLM level, the lowest level of the length LT including in the present study, and decreased in frequency as the length LT levels increased in sophistication. The use of the complexity intuition, even by students operating predominantly at the highest level of the current length LT, the ALM level, suggests that increased conceptual and procedural knowledge for length measurement does not preclude the evocation of an intuition that is not mathematically productive. Therefore, the findings of this study support recommendations of prior research (Chiu, 1996) that instructional experiences should be designed to elicit students' intuitions and position them to confront and make sense of those intuitions using other intuitions and analytical thinking.

Findings related to students' strategies for comparing curves using nonstandard units (Interview Tasks 3, 4, 5 6B, and 8B) and measuring curves with a standard ruler (Interview Tasks 9 and 10) show that these tasks have the potential to provide an instructionally fruitful context for addressing measurement from both a science perspective and a mathematical perspective (Osborne, 1976). In science, measurement is a range of numbers; it is a process and a skill used for the purposes of building models of reality and subsequently testing the truths of those models of reality. In mathematics, however, measurement is a single number, an entity; the test of truth for measurement from a mathematical perspective involves the correctness of reasoning.

More specifically, the findings reported here suggest that measurement tasks involving determining curve length (Interview Tasks 3, 4, 5, 6B, 8B, 9, and 10) have the potential to address key measurement concepts as outlined in a learning progression in science education, the LP for AMTM (NRC, 2007; Smith, Wiser, Anderson, and Krajcik, 2006). According to the LP for AMTM students in Grades 3 through 5 should understand that measurements could be more or less precise and that there is always some error in measurement, and students in Grades 6 through 8 should learn that sources of measurement error can be examined and quantified. In the present study, participants within and across all four length LT groups exhibited instances of acknowledging that they had under- or overestimated the length of a curve. However, students within and across all four length LT groups also made claims that their way of determining the length of a curve was not an under- or overestimate. This suggests that students in Grade 4, 6, 8, and 10 could benefit from an instructional activity in which they measure curves with standard or nonstandard units (Tasks 3, 4, 5, 6B, 9, and 10), share their strategies for measuring the curves, and engage in a follow-up discussion about sources contributing to the error involved with their ways of measuring the curves and how they could increase or decrease the precision of their measurements.

Furthermore, Osborne (1976) noted that determining the length of a curve "is a step beyond most school mathematics" (p. 24) because the solution involves limit processes, or the additivity principle extended to allow for the addition of infinitely many segments. However, the results reported here indicate that measurement tasks involving determining the length of a curve (Interview Tasks 3, 4, 5, 6B, 8B, 9, and 10) have potential instructional value for eliciting and discussing measurement from a mathematical perspective (Osborne, 1976) using informal limit arguments, an approach that has been recommended for secondary students in the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010). In the present study, when measuring a curve with a nonstandard unit (Tasks 3, 4, 5, 6B, and 8B), participants exhibited 20 instances of fracturing the nonstandard unit around the entire curve (see Table 16, Chapter 4). These instances occurred most often in Grades 6, 8, and 10. This suggests that by middle school, in an instructional setting, students may be ready to use and make sense of informal limit arguments by discussing processes in which a curve is represented by increasingly large numbers of segments of decreasing lengths to decrease the error in measuring and approach a true length of the curve.

Implications for Future Research

The present study made use of a written length LT-based assessment. Given that such an instrument could play an important role in meeting recommendations for extending and validating LTs (Daro, Mosher, & Corcoran, 2011), future research should be aimed at refining these assessment items included here to develop a reliable and valid LT-based assessment instrument. Some of the items on the assessment included in this study were less reliable than others in terms of assessing the concepts and processes they were designed to address. For example, between 30 and 45% of students' responses within each of Grades 4, 6, 8, and 10 were coded as "No Claim" for Task 5, which was designed to elicit concepts and processes at the ICPM and ALM levels of the length LT (although it was also designed to be accessible to students at the CLM and CRM levels). This suggests that Task 5 may not be a valid task for eliciting students' thinking at the CLM, CRM, ICPM, and ALM levels. Similarly, Task 7 yielded codes of "No Claim" in most instances, and was not considered as part of the task-by-task analysis of the assessment. With a key affordance of the ease of administration of paper-pencil instruments, future research should include iterations of design cycles that include pilot and design work aimed at revising these items, followed by administering the revised items and examining reliability and validity of the revised instrument.

The present study established interactions among intuitive and analytical thinking for path length with concept growth along an LT for length measurement; however, given the exploratory nature of this study, the changeability of intuitions and analytical strategies for children at each of the relevant LT levels was not explored. Future studies should extend this work by examining the (a) perturbability of intuitive and analytical thinking for students operating at the levels of the length LT included in the present study: the CLM, CRM, ICPM, and ALM levels, and (b) the types of instructional interventions that can support changes in students' intuition use. This research should also examine the impact of causing change in students' ways of using intuitions for path length on their level of sophistication for length measurement, or vice versa.

Furthermore, future research should emphasize the nativist component of hierarchic interactionalism, and examine the intuitive and analytical thinking for rectilinear and curvilinear paths for children at lower levels of the length LT that were not included in the present study. Such a study could show how intuitions and analytical strategies might be formed as children transition from the initial level of the length LT at which they recognize length as a quantity, at the Length Quantity Recognizer level (LQR), to simultaneously developing levels at which children begin to compare objects by length directly and indirectly, at the Length Comparer Level (LC) and develop the implicit concept that an object can be composed of smaller objects (EE), to the level at which unit iteration develops (LURR). This study could shed light on how direct and indirect comparison for length measurement develop along with subsequent length measurement levels over time, which is still an open question for researchers in mathematics education (Battista, 2006; Clements et al, in press), and how intuitive and analytical thinking play a role in that development.

Finally, results from the present study suggest that measurement tasks that involve determining curve length using nonstandard straight units or standard units, such as a ruler, have potential instructional value from both a scientific and mathematical perspective. Further research is needed to explore the instructional affordances of such tasks for eliciting students' thinking about the role and sources of error in measurement, as recommended in the LP for AMTM for elementary and middle school students. In addition, future studies should investigate whether the tasks involving curves in the present study could have instructional value for eliciting and supporting students' use of informal limit arguments to make sense of measurement from a mathematical perspective.

REFERENCES

Barrett, J. E., & Clements, D. H. (2003). Quantifying path length: Fourth-grade children's developing abstractions for linear measurement. *Cognition and Instruction*, *21*(4), 475–520.

Barrett, J. E., & Eames, C. L. (2013). *Development and Use of Learning Trajectories as Tools: Research, Curriculum, and Professional Development.* Session presented at the Center for Mathematics, Science, and Technology, Illinois State University, Normal, IL, January 15, 2013.

Barrett, J. E., Clements, D. H., Klanderman, D., Pennisi, S. J., & Polaki, M. V. (2006). Students' coordination of geometric reasoning and measuring strategies on a fixed perimeter task: Developing mathematical understanding of linear measurement. *Journal for Research in Mathematics Education*, *42*, 637–650.

Barrett, J. E., Cullen, C. J., Sarama, J. Clements, D. H. Klanderman, D., Miller, A. L., Rumsey, C. (2011). Children's unit concepts in measurement: A teaching experiment spanning grades 2 through 5. *ZDM*, *43*(5), 637–650.

Barrett, J. E., Sarama, J., Clements, D. H., Cullen, C., McCool, J., Witkowski-Rumsey, C., & Klanderman, D. (2012). Evaluating and improving a learning trajectory for linear measurement in elementary grades. *Mathematical Thinking and Learning*, *14*, 28–54.

Battista, M. T. (2006). Understanding the development of students' thinking about length. *Teaching Children Mathematics*, 13(3), 140–146.

Beck, P., Eames, C. L., Cullen, C. J., Barrett, J. E., Clements, D. H., & Sarama, J. (2014). Linking children's knowledge of length measurement to their use of double number lines. In Nicol, C., Liljedahl, P., Oesterle, S., & Allan, D. (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education*, Vol. 2. (pp. 105–112). Vancouver, CA: PME.

Carpenter, T. P., & Lewis, R. (1976). The development of the concept of a standard unit of measure in young children. *Journal for Research in Mathematics Education*, *7*(1), 1976, 53–58.

Chiu, M. M. (1996). Exploring the origins, uses, and interactions of student intuitions: Comparing the lengths of paths. *Journal for Research in Mathematics Education, 27*(4), 478–504.

Clarke, D., Cheeseman, J., McDonough, A., & Clarke, B. (2003). Assessing and developing measurement with young children. In D. H. Clements & Bright (Eds.) *Learning and Teaching Measurement*: 2003 Yearbook (pp. 68–80). Reston, VA: National Council of Teachers of Mathematics.

Clements, D. H., Barrett, J. E., Sarama, J., Cullen, C. J., Van Dine, D. W., Eames, C. L., Miller, A. L., Kara, M., Klanderman, D., & Vukovich, M. (in press). Length – A summary report. In J. E. Barrett, J. Sarama, & D. H. Clements, (Eds.), *A longitudinal account of children's knowledge of measurement* (JRME Monograph No. X). Reston, VA: National Council of Teachers of Mathematics

Clements, D. H., Battista, M. T., Sarama, J., Swaminathan, S., & McMillen, S. (1997). Students' development of length measurement concepts in a logo-based unit on geometric paths. *Journal for Research in Mathematics Education, 28,* 49–70.

Clements, D. H., & Sarama, J. (2007). Early childhood mathematics learning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 461–555). Charlotte, NC: Information Age Publishing.

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. Retrieved from http://www.corestandards.org.

Corbin, J., & Strauss, A. (Eds.). (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Sage.

Cullen, C. J. (2009). *A comparative analysis: Two representational models for units of length.* (Order No. 3388933, Illinois State University). *ProQuest Dissertations and Theses,* 144-n/a. Retrieved from http://search.proquest.com/docview/304899892?accountid=11578. (304899892).

Cullen, C. J., Miller, A. L., Witkowski-Rumsey, C., Barrett, J. E., & Sarama, J. (2011). *Area hypothetical learning trajectory: Relating square units to non-rectilinear regions*. Paper presented at the Annual Meeting of the American Educational Research Association: New Orleans, LA, April 8 – 12, 2011.

Cross, C. T., Woods, T. A., & Schweingruber, H. (2009). *Mathematics learning in early childhood*. Washington DC: The National Academies Press.

Daro, P., Mosher, F. A., & Corcoran, T. (2011). *Learning trajectories in mathematics: A* *foundation for standards, curriculum, assessment, and instruction* (CPRE Report No. RR-68). Philadelphia, PA: Pearson.

Dickson, L. Brown, M., & Gibson, O. (1984). *Children Learning Mathematics: A Teachers Guide to Recent Research*. London: Holt Rinehart Winson.

Ellis, S., Siegler, R. S., & Van Voorhis, F. E. (2003, April). *Developmental changes in children's understanding of measurement procedures and principles*. Paper presented at the Biennial Meeting of the Society for Research in Child Development, Tampa, FL.

Fischbein, E. (1987*). Intuition in Science and Mathematics: An Educational Approach*. Dordrecht, Holland: Reidel Publishing Company.

Goldin, G. A. (2000). A scientific Perspective on structured task-based interviews in mathematics education research. In A. E. Kelly, & R. A. Lesh, *Handbook of research design in mathematics and science education* (pp. 517-546). Mahwah, NJ: Lawrence Erlbaum.

Grugnetti L., & Rizza, A., Marchini, C. (2007). A lengthy process for the establishment of the concept of limit starting form pupils' pre-conceptions. *Far East Journal of Mathematics Education*, *1*(1), 1–32.

Hiebert, J. (1981). Cognitive development and learning linear measurement. *Journal for Research in Mathematics Education, 12*(3), 197–211.

Horvath, J. K., & Lehrer, R. (2000). The design of a case-based hypermedia teaching tool. *International Journal of Computers for Mathematical Learning*, *5*(2), 115-141.

Kara, M. (2013). *Students' reasoning about invariance of volume as a quantity*. (Order No. 3592420, Illinois State University). *ProQuest Dissertations and Theses*, 332. Retrieved from http://search.proquest.com/docview/1436986351?accountid=11578. (1436986351).

Kloosterman, P., Warfield, J., Wearne, D., Koc, Y., Martin, G., & Strutchens, M. (2004). Knowledge of mathematics and perceptions of learning mathematics of fourth grade students. In p. Kloosterman & F. K. Lester (Eds.), *Results and interpretations of the 2003 mathematics assessment of the National Assessment of Educational Progress*. Reston, VA: National Council of Teachers of Mathematics.

Kosslyn, S. M., Pick, H. L., Jr., & Fariello, G. R. (1974). Cognitive maps in children and men. *Child Development*, *45*, 707–716.

Lehrer, R. (2003). Developing understanding of measurement. In J. Kilpatrick, W. G. Martin, and D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics* (pp. 179 – 192). Reston, VA: National Council of Teachers of Mathematics.

Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study of children's reasoning about space and geometry. In R. Lehrer & D. Chazan (Eds.) *Designing learning environments for developing understanding of geometry and space* (Vol. 1, pp. 137 – 167). Mahwah, NJ: Lawrence Erlbaum Associates.

Lincoln, Y. S., & Guba, E. G. (2000). *Naturalistic inquiry*. Thousand Oaks, CA: Sage.

Luria, S. M., Kinney, J. S., & Weissman, S. (1967). Distance estimates with "filled" and "unfilled" space. *Perceptual and Motor Skills*, *24*, 1007–1010.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation*. San Francisco, CA: John Wiley & Sons.

Nunes, T., & Bryant, P. E. (1996). *Children doing mathematics*. Cambridge, MA: Blackwell.

Osborne, A. R. (1976). Mathematical distinctions in the teaching of measure. In D. Nelson (Ed.), *Measurement in school mathematics* (pp. 11–34). Reston, VA: National Council of Teachers of Mathematics.

Pengelly, H. & Ranking, L. (1995). *Linear Measurement: Children's Developing Thoughts*. Adelaide, South Australia: Department of Education.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry* (E. A. Lunzer, Trans. 1981 Basic Books, Inc. ed.). New York, NY: W. W. Norton & Company.

Pressey, A. W. (1974). Age changes in the Ponzo and filled-space illusions. *Perception & Psychophysics*, *15*, 315–319.

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York, NY: Routledge.

Smith, C. L., & Wiser, M., Anderson, C. W., & Krajcik, J. (2006). Implications of research on children's learning for standards and assessment: A proposed learning progression for matter and atomic-molecular theory. *Measurement, 14*, 1–98.

Sowder, J. T., Wearne, D., Martin, G., & Strutchens, M. (2004). What grade 8 students know about mathematics: Changes over a decade. In P. Kloosterman & F. Lester (Eds.), *The 1990 through 2000 mathematics assessment of the National*

Assessment of Educational Progress: Results and Interpretations. (pp. 105 – 143). Reston, VA: National Council of Teachers of Mathematics.

Stephan, M., & Clements, D. H. (2003). Linear and area measurement in prekindergarten to grade 2. In D. H. Clements (Ed.), *Learning and teaching measurement: 2003 yearbook* (pp. 3–16). Reston, VA: National Council of Teachers of Mathematics.

Stephan, M., Bowers, J., Cobb, P., & Gravemeijer, K. (2004). *Supporting students' development of measuring conceptions: Analyzing students' learning in social context (Journal for Research in Mathematics Education monograph #12).* Reston, VA: National Council of Teachers of Mathematics.

Thorndyke, P. W. (1981). Distance estimation from cognitive maps. *Cognitive Psychology*, *23*, 526–550.

Traub, G. (1984). *The development of the mathematical analysis of curve length from Archimedes to Lebesgue* (Unpublished doctoral dissertation). New York University, NY.

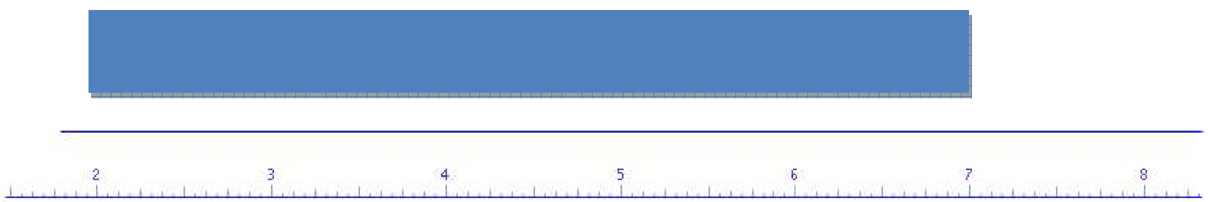
Wilson, P. S. & Rowland, R. (1993). Teaching measurement. In R. J. Jenson (Ed.), *Research ideas for the classroom: Early childhood mathematics* (pp. 171–194). New York, NY: Macmillan.

APPENDIX A

WRITTEN LENGTH LT-BASED ASSESSMENT

Name: ___________________ Grade: __________________
Teacher: _________________ School: __________________

1.

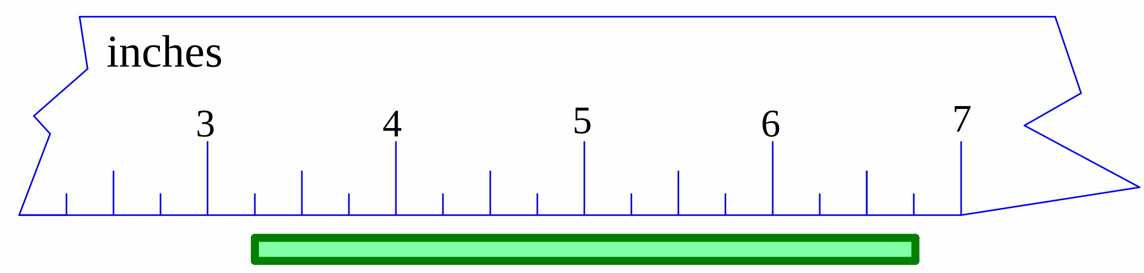


Using the drawing of a part of a ruler as a guide, measure the strip of paper shown above it. How many inches long is the strip?

Write your answer on the line.

____

2.

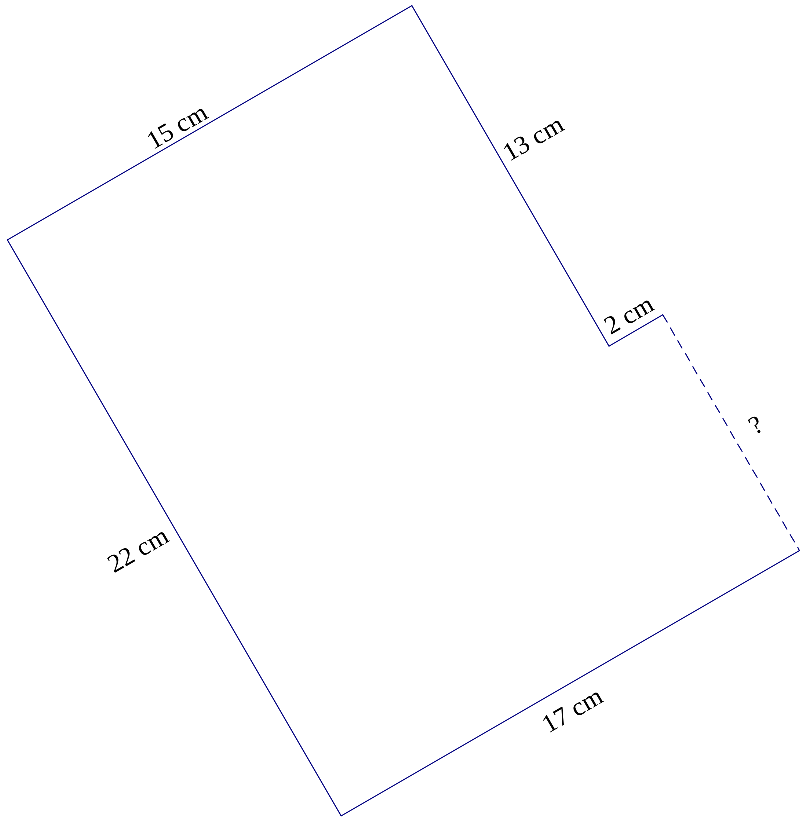


This is a picture of a rod just below a broken section of a ruler. Use this picture to measure the length of the rod. How long is the rod?

Write your answer on the line.

____

3.

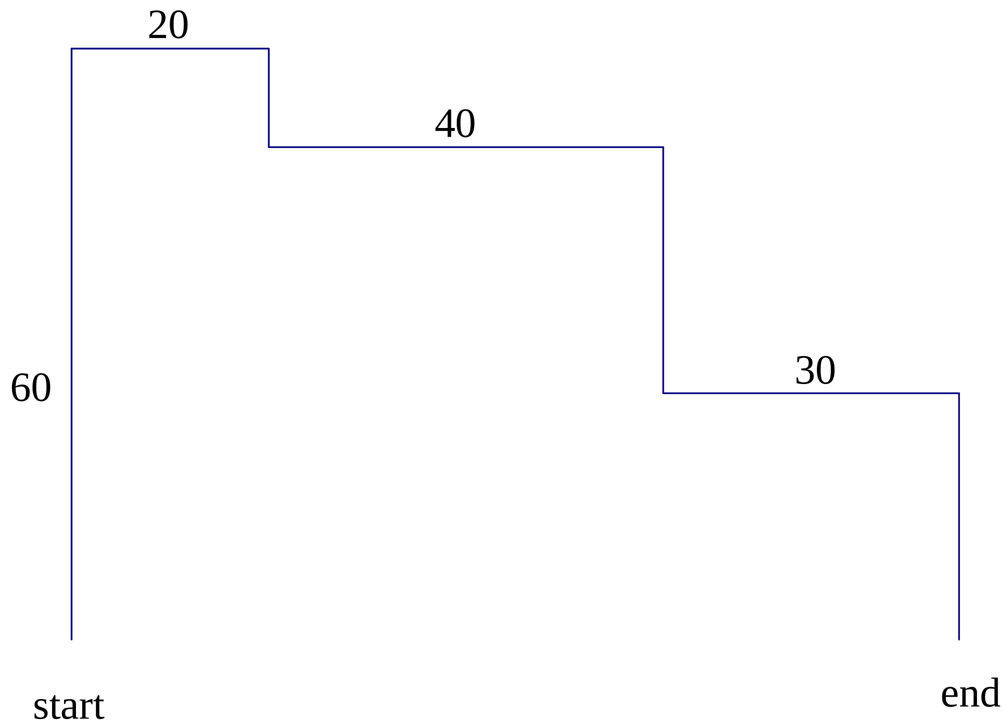


Find the measure of the missing side length.

Write your answer on the line.

____

4.



Find the length of the total path, from start to end, shown above.

Write your answer on the line.

____

5.

Imagine making an L-shaped path from a string that is 10 cm long.

a. How many different L-shaped paths would you be able to form in all?

b. Use the space below to explain how you got your answer and why you think your answer is correct.

6.

a. Use the space below to sketch **two different** rectangles, each having a perimeter of 2 inches. For each of your rectangles, label the lengths of all four sides.

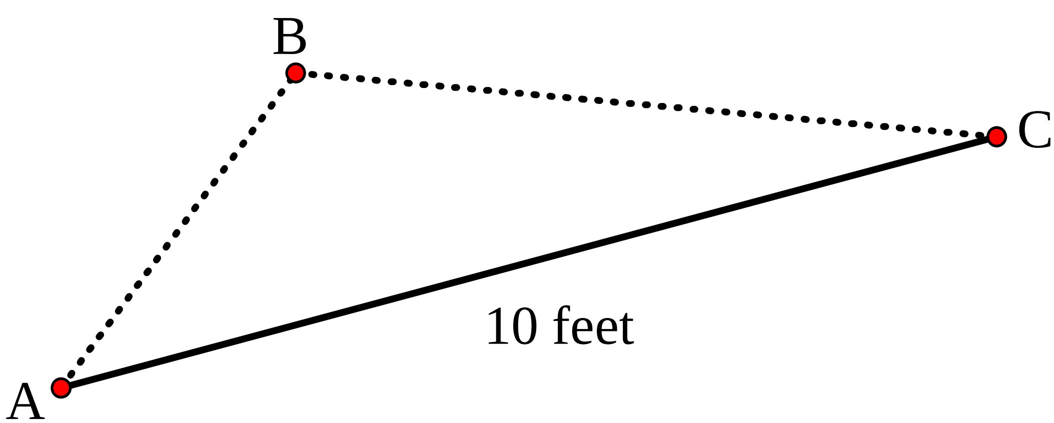
b. How many **more** rectangles have a perimeter of 2 inches?

____

7.

You need to bury a wire in your backyard that connects points A and C. One option is to run a 10-foot wire directly from points A and C, which is indicated by the solid line in the picture below. Another option is to run a wire from point A to C through point B, which is indicated by the dotted line.

We know that points A and C are 10 feet apart. However, no one measured the length of the path from A to C through point B (the dotted line).



a. How *long* you think the wire will need to be to connect points A and C through B?____

b. Use the space below to explain how you got your answer for part *a*, and why you think your answer is correct.

c. How much wire will you *buy* so that you can be sure you have enough to connect points A and C through B?

____

d. Use the space below to explain how you got your answer for part *c*, and why you think your answer is correct.

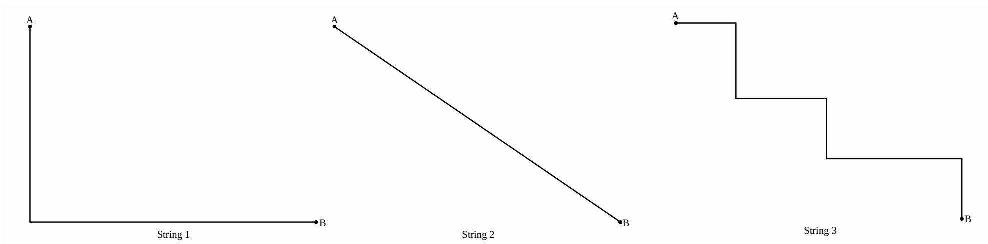
APPENDIX B

INTERVIEW PROTOCOLS

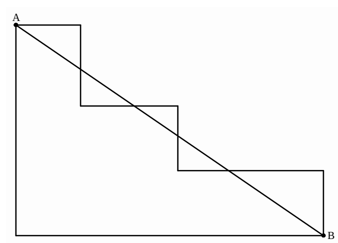
Interview Set 1

Task 1: Simple Rectilinear Bent Path Comparison Task (Chiu, 1996)

Show the student the three "strings" each printed on a separate transparency.



Stage 1 (posing the problem): Overlap all three of the "strings" to show that they connect the same points A and B.



Here are three different ways that points A and B could be connected with string. (Separate the three "strings" and place them in front of the student in a row.) *Compare Strings 1, 2, and 3 by their lengths.*
Nondirective follow-up: (If the student orders the strings.) *Tell me about your order.*

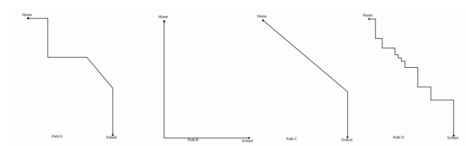
Stage 2 (minimal heuristic suggestion) If the student does not immediately answer, provide the student with a marker and ask: *Can you move them or use the marker to write while you think about comparing the strings by length?*

Stage 3 (guided use of a heuristic suggestion)
Do you think all of the strings are different lengths or are any the same length?

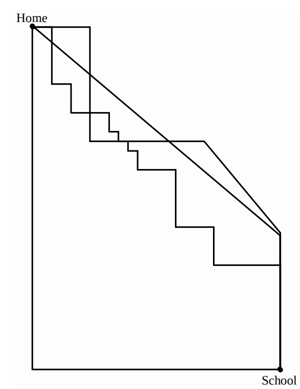
Stage 4 (exploratory and metacognitive)
Why is string ____ the shortest?
What is it about string ____ that makes you think it is the shortest?
Why does ______ (the feature of the string described by the student) *make string ____ the shortest?*
Why is string ____ the longest?
What is it about string _____ that makes you think it is the longest?
Why does ______ (the feature of the string described by the student) *make string ____ the longest?*

Task 2: Complex Rectilinear Bent Path Comparison Task (Chiu, 1996)

Show the student the four "paths" each printed on a separate transparency.



Stage 1 (posing the problem): Overlap all four of the "paths" to show that they all connect "home" to "school."
Here are four different paths that someone I know sometimes takes from home to school. (Separate the four "paths" and place them in front of the student in a row.) *Compare Paths A, B, C, and D by their lengths.*



Nondirective follow-up: (If the student orders the paths.) *Tell me about your order.*

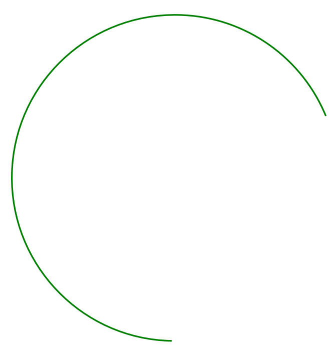
Stage 2 (minimal heuristic suggestion)
If the student does not immediately answer, provide the student with a marker and ask, *Can you move them or use the marker to write while you think about putting the paths in order by their lengths from shortest to longest?*

Stage 3 (guided use of a heuristic suggestion)
Do you think all of the paths are different lengths or are any that are the same length?

Stage 4 (exploratory and metacognitive)
Why is path ____ the shortest?
What is it about path ____ that makes you think it is the shortest?
Why does ______ (the feature of the path described by the student) *make path ____ the shortest?*
Why is path _____ longer than path _____? (the 3rd and 2nd paths in the student's ranking)
What is it about path _____ that makes you think it longer than path _____?
Why does ______ (the feature of the path described by the student) *make path ____ the longer than path _____?*
Why is path ____ the longest?
If the student's answer is not clear: *What is it about path _____ that makes you think it is the longest? Why does ______* (the feature of the path described by the student) *make path ____ the longest?*

Task 3: Compare Curve to Stick (Clements et al., in press)

Provide the piece of paper with the following image, a four-inch wooden stick, and a pen.



Stage 1 (posing the problem)
Say: *Compare the length of this curved path* (trace finger around the path) *to this stick.*
***Nondirective follow-up**:* If the student provides a qualitative comparison (i.e. says the curved path is longer) ask,
how much longer?

<u>Stage 2</u> <u>(minimal heuristic suggestion)</u>
If the student does not immediately answer: *Which one is longer the curved path or the stick?*

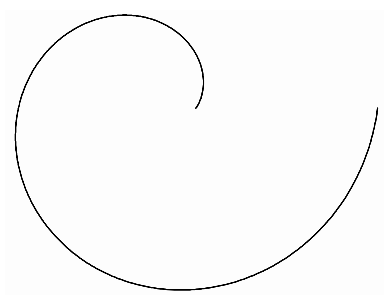
<u>Stage 3</u> <u>(guided use of a heuristic suggestion)</u>
Could you use the path and the stick to show me how much longer? Show me where the _____ are.

<u>Stage 4</u> <u>(exploratory and metacognitive)</u>
Explain how you thought about comparing the curved path to the stick.
Make a record of how you compared the curved path to the stick by drawing to show how you laid the stick.

<u>Follow up</u>
Is your answer an over- or an under-estimate for the length of this curve?
How do you know?

Task 4: Compare Curve to Stick (Clements et al., in press)

Provide the piece of paper with the following image, a four-inch wooden stick, and a pen.



<u>Stage 1</u> <u>(posing the problem)</u>
<u>Say:</u> *Compare the length of this curved path* (trace finger around the path) *to this stick.*
Nondirective follow-up: If the student provides a qualitative comparison (i.e. says the curved path is longer) ask, *how much longer?*

<u>Stage 2</u> <u>(minimal heuristic suggestion)</u>
If the student does not immediately answer: *Which one is longer the curved path or the stick?*

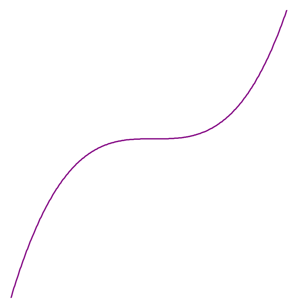
<u>Stage 3</u> <u>(guided use of a heuristic suggestion)</u>
Could you use the path and the stick to show me how much longer? Show me where the _____ are.

<u>Stage 4</u> <u>(exploratory and metacognitive)</u>
Explain how you thought about comparing the curved path to the stick.
Make a record of how you compared the curved path to the stick by drawing to show how you laid the stick.

<u>Follow up</u>
Is your answer an over- or an under-estimate for the length of this curve?
How do you know?

Task 5: Compare Curve to Stick (Clements et al., in press)

Provide the piece of paper with the following image, a four-inch wooden stick, and a pen.



<u>Stage 1</u> <u>(posing the problem)</u>
<u>Say:</u> *Compare the length of this curved path* (trace finger around the path) *to this stick.*
Nondirective follow-up: If the student provides a qualitative comparison (i.e. says the curved path is longer) ask, *how much longer?*

<u>Stage 2</u> <u>(minimal heuristic suggestion)</u>
If the student does not immediately answer: *Which one is longer the curved path or the stick?*
<u>Stage 3</u> <u>(guided use of a heuristic suggestion)</u>
Could you use the path and the stick to show me how much longer? Show me where the _____ are.

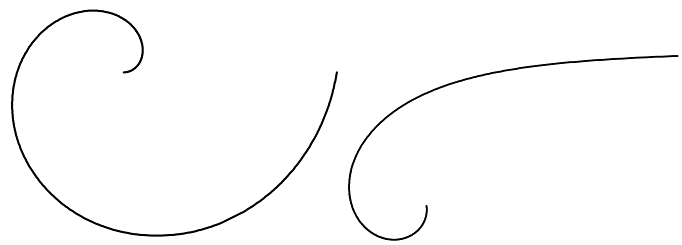
<u>Stage 4</u> <u>(exploratory and metacognitive)</u>
Explain how you thought about comparing the curved path to the stick.
Make a record of how you compared the curved path to the stick by drawing to show how you laid the stick.

<u>Follow up</u>
Is your answer an over- or an under-estimate for the length of this curve?
How do you know?

Interview Set 2

Tasks 6A and 6B: Compare Two Curves (Clements et al., in press)

Provide the two pieces of paper with each of the following curved paths.



__**Task 6A: Stage 1** (posing the problem)__
__Say:__ *Compare the length of this curve* (trace finger around curve) *to the length of this curve (trace finger around curve).*
Nondirective follow-up*: Tell me how you thought about comparing these curves*

__**Task 6A: Stage 2** (minimal heuristic suggestion)__
If the student does not immediately answer: *Do you think these curves are different lengths or are they the same length?*

__**Task 6A: Stage 3** (guided use of a heuristic suggestion)__
Could you point and show me on the curves?

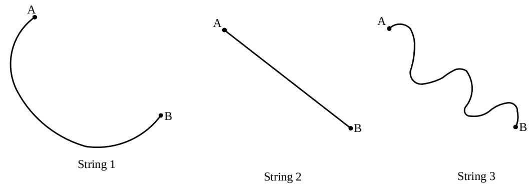
__**Task 6A: Stage 4** (exploratory and metacognitive)__
Why is this curve longer than this curve?
If the student's answer is not clear: *What is it about this curve _____ that makes you think it longer than that curve?*
Why does ______ (the feature of the curve described by the student) *make this curve longer than that curve?*

__**Task 6B: Using the stick to check**__
Please use this stick to help you check. Explain how you thought about comparing the curved path to the stick.
Make a record of how you compared the curved path to the stick by drawing to show how you laid the stick.

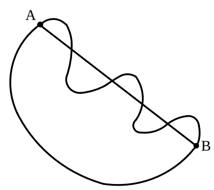
__**Follow up**__
Is your answer an over- or an under-estimate for the length of this curve?
How do you know?

Task 7: Compare 3 Curves

Show the student the three "strings" each printed on a separate transparency.



Here are three different ways that points A and B could be connected with string. (Separate the three "strings" and place them in front of the student in a row.) *Compare Strings 1, 2, and 3 by their lengths.*



Nondirective follow-up: (If the student orders the strings.) *Tell me about your order.*

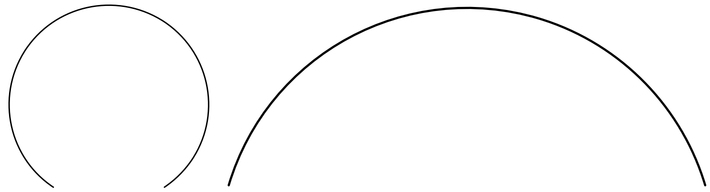
<u>**Stage 2** (minimal heuristic suggestion)</u>
If the student does not immediately answer, provide the student with a marker and ask: *Can you move them or use the marker to write while you think about comparing the strings by length?*

<u>**Stage 3** (guided use of a heuristic suggestion)</u>
Do you think all of the strings are different lengths or are any the same length?

<u>**Stage 4** (exploratory and metacognitive)</u>
Why is string ____ the shortest?
What is it about string ____ that makes you think it is the shortest?
Why does ______ (the feature of the string described by the student) *make string ____ the shortest?*
Why is string ____ the longest?
What is it about string _____ that makes you think it is the longest?
Why does ______ (the feature of the string described by the student) *make string ____ the longest?*

Tasks 8A and 8B: Compare Two Curves (Clements et al., in press)

Provide the two pieces of paper each with one of the following curves.



Task 8A: Stage 1 (posing the problem)
Say: *Compare the length of this curve* (trace finger around curve) *to the length of this curve (trace finger around curve).*
Nondirective follow-up: *Tell me how you thought about comparing these curves*

Stage 2 (minimal heuristic suggestion)
If the student does not immediately answer: *Do you think these curves are different lengths or are they the same length?*

Stage 3 (guided use of a heuristic suggestion)
Could you point and show me on the curves?

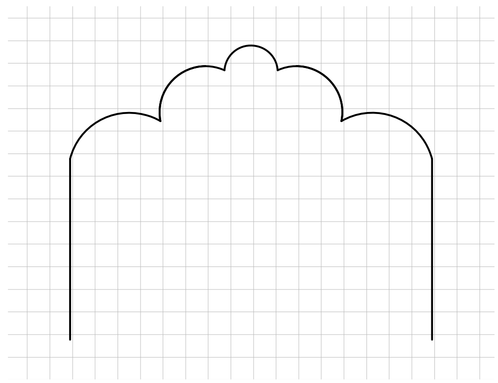
Stage 4 (exploratory and metacognitive)
Why is curve ____ the shortest?
If the student's answer is not clear: *What is it about curve ____ that makes you think it is the shortest? Why does ______* (the feature of the curve described by the student) *make curve ____ the shortest?*

Task 8B: Using the stick to check
Please use this stick to help you check. Explain how you thought about comparing the curved path to the stick.
Make a record of how you compared the curved path to the stick by drawing to show how you laid the stick.
Please use this stick to help you check. Explain how you thought about comparing the curved path to the stick.
Make a record of how you compared the curved path to the stick by drawing to show how you laid the stick.

Follow up
Is your answer an over- or an under-estimate for the length of this curve?
How do you know?

Task 9: Measure the Outline of a Doorway (Grugnetti, Rizza, & Marchini, 2007)

Provide the piece of paper with the following image, a standard ruler, and a pen.



<u>**Stage 1** (posing the problem)</u>
Do you know what a blueprint is?
This is a drawing of the outline of a doorway like on a blueprint, but it has no measurements. Please measure the outline of this doorway in the most precise way that you can using this ruler.

Nondirective follow-up: *Tell me about your way of measuring the outline of this doorway.*

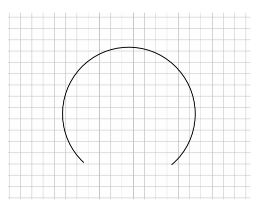
<u>**Stage 2** (minimal heuristic suggestion)</u>
If the student does not immediately answer ask, *How do you think someone might try to get as close to the length of the outline of this doorway as they can using this ruler?*

<u>**Stage 3** (guided use of a heuristic suggestion)</u>
Do you think people might use different methods to get very close to the actual length of the outline of this doorway?

<u>**Stage 4** (exploratory and metacognitive)</u>
Explain how you measured it in the most precise way.

Task 10: Measure the Outline of a Rounded Doorway (Grugnetti, Rizza, & Marchini, 2007)

Provide the piece of paper with the following image, a standard ruler, and a pen.



Stage 1 (posing the problem)
Here is another outline of a doorway from a blueprint. Please measure the outline of this doorway in the most precise way that you can using this ruler. ***Nondirective follow-up:*** *Tell me about your way of measuring the outline of this doorway.*

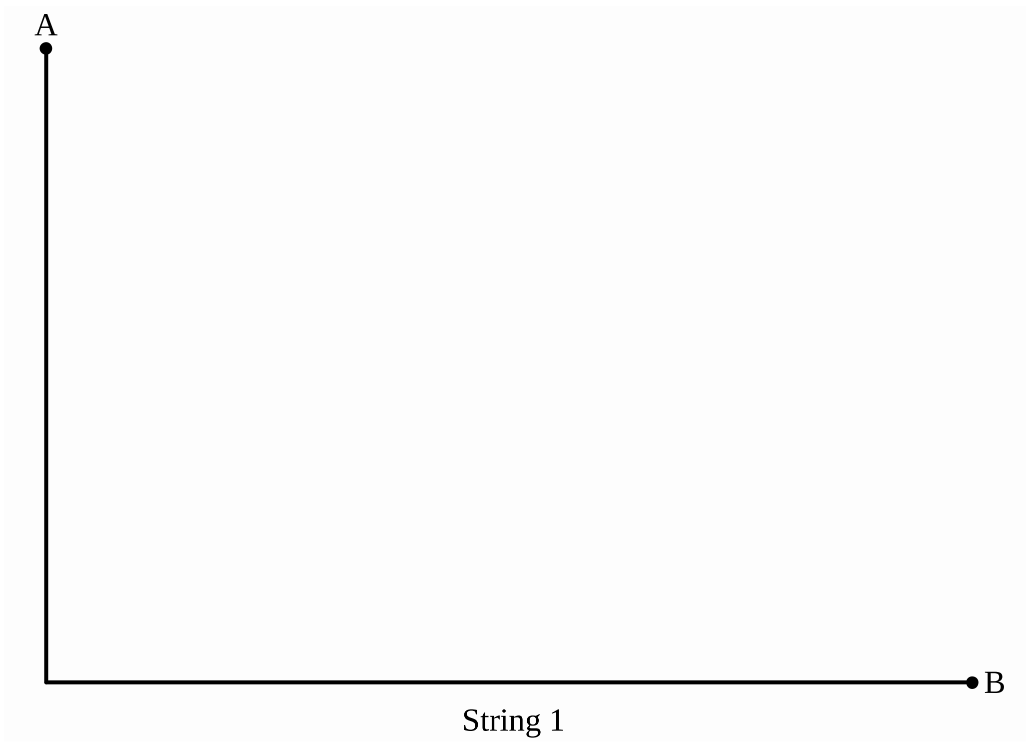
Stage 2 (minimal heuristic suggestion)
If the student does not immediately answer ask, *How do you think someone might try to get as close to the length of the outline of this doorway as they can using this ruler?*

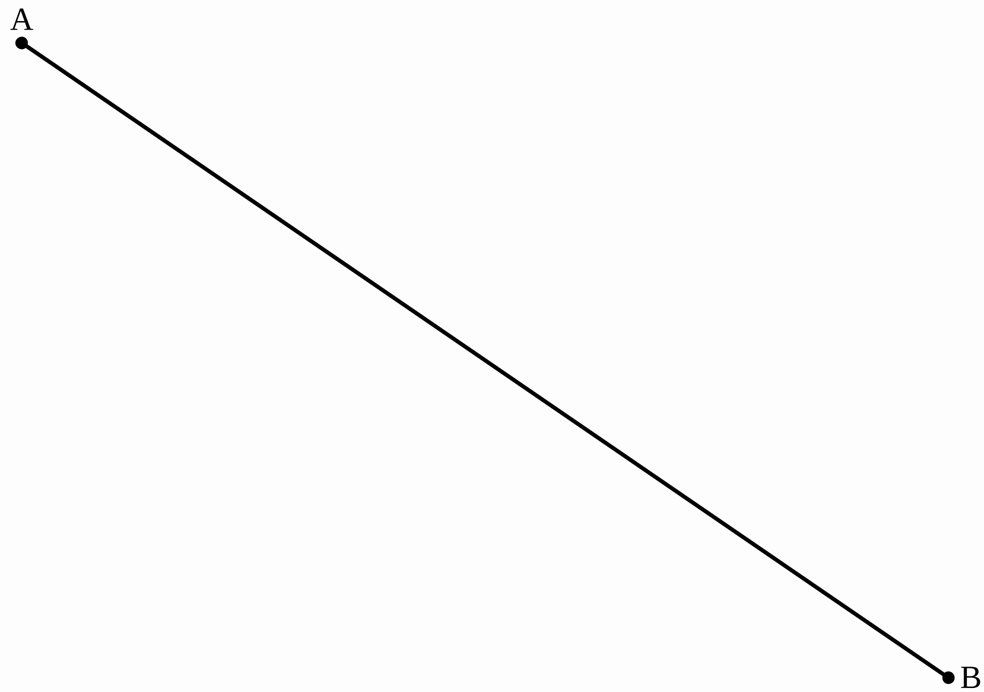
Stage 3 (guided use of a heuristic suggestion)
Do you think people might use different methods to get very close to the actual length of the outline of this doorway?

Stage 4 (exploratory and metacognitive)
Explain how you measured it in the most precise way.

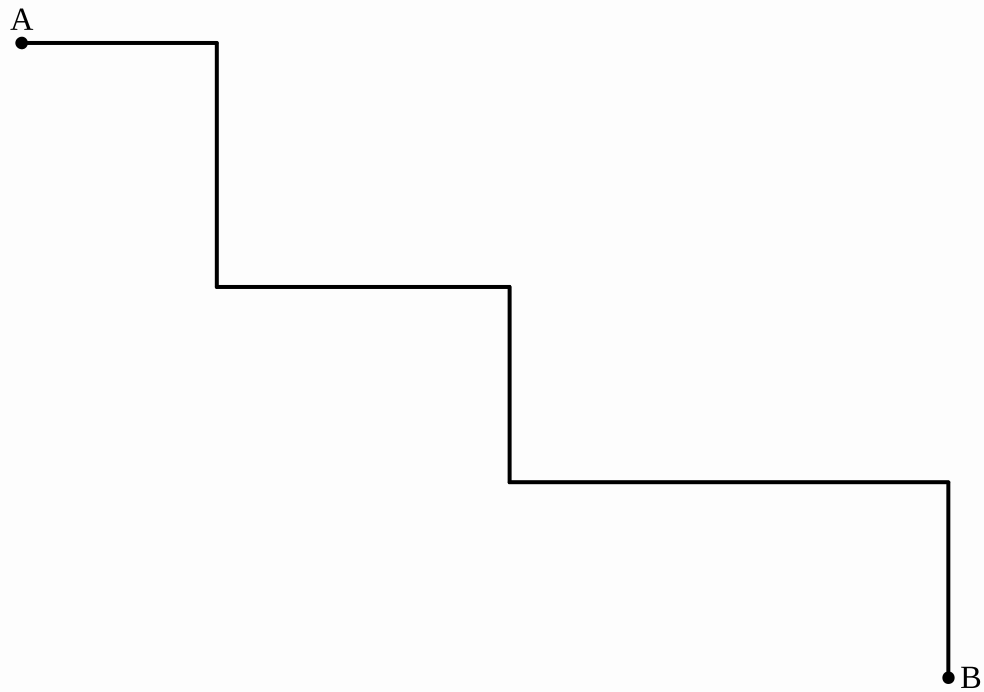
APPENDIX C

IMAGES SHOWN DURING INTERVIEWS

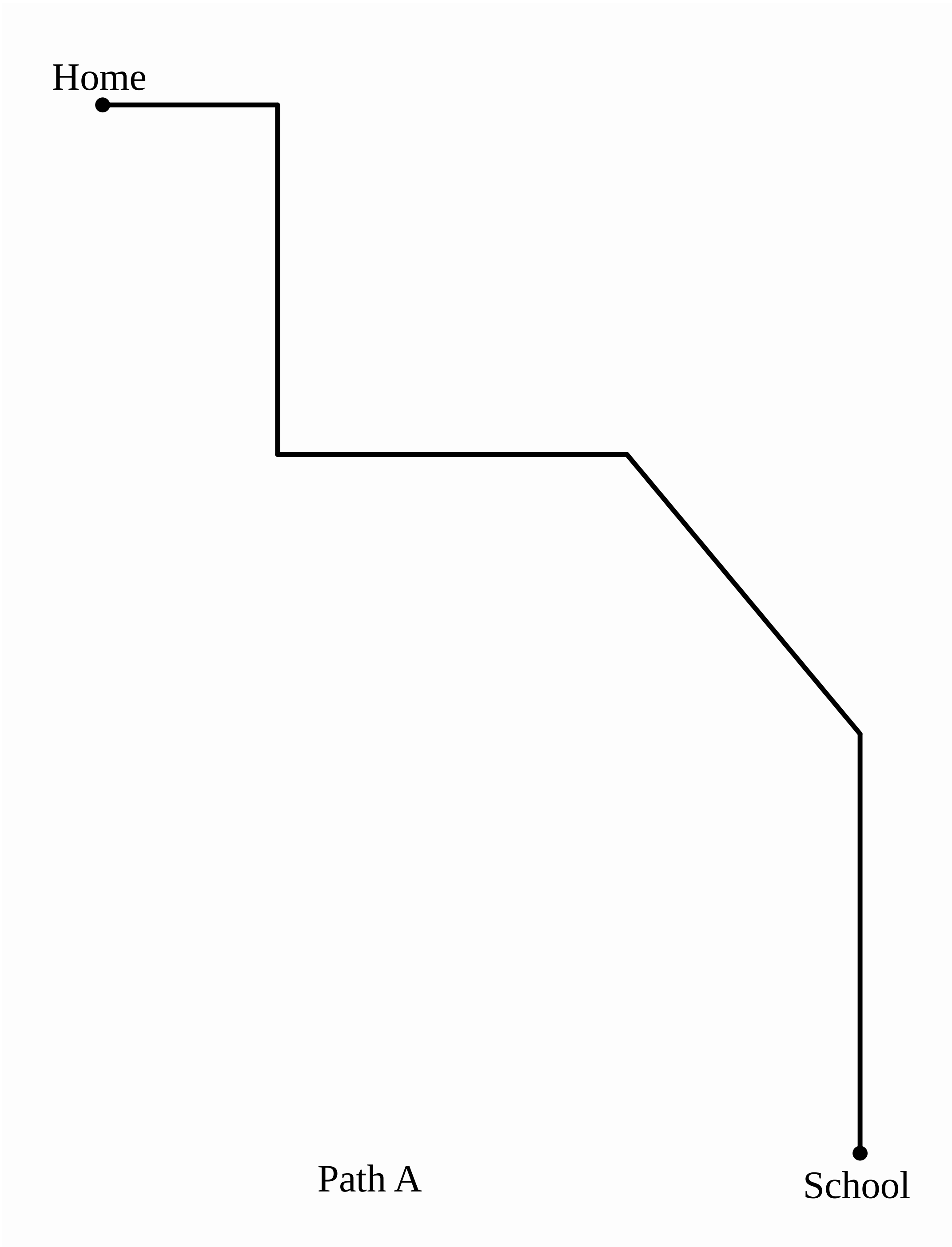


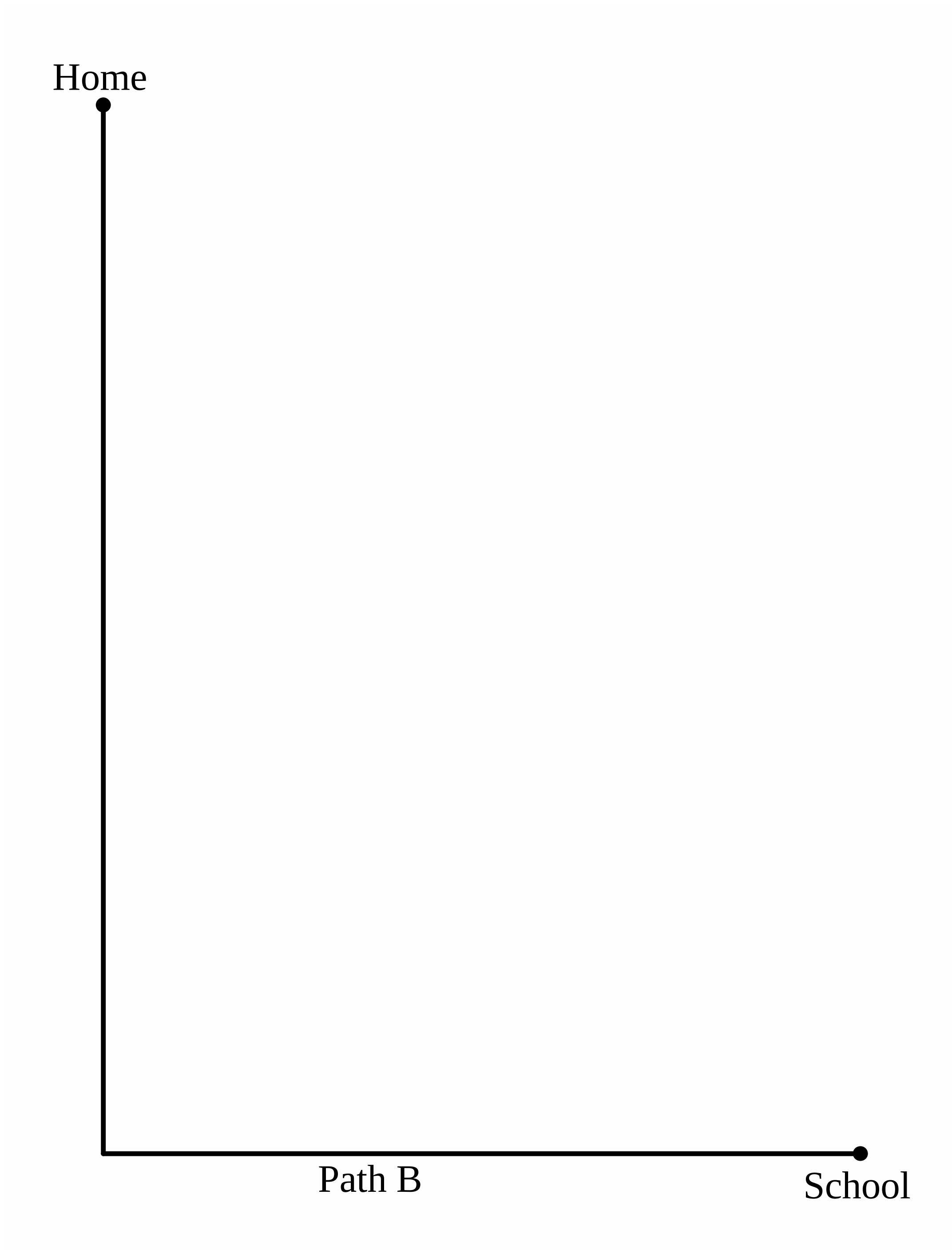


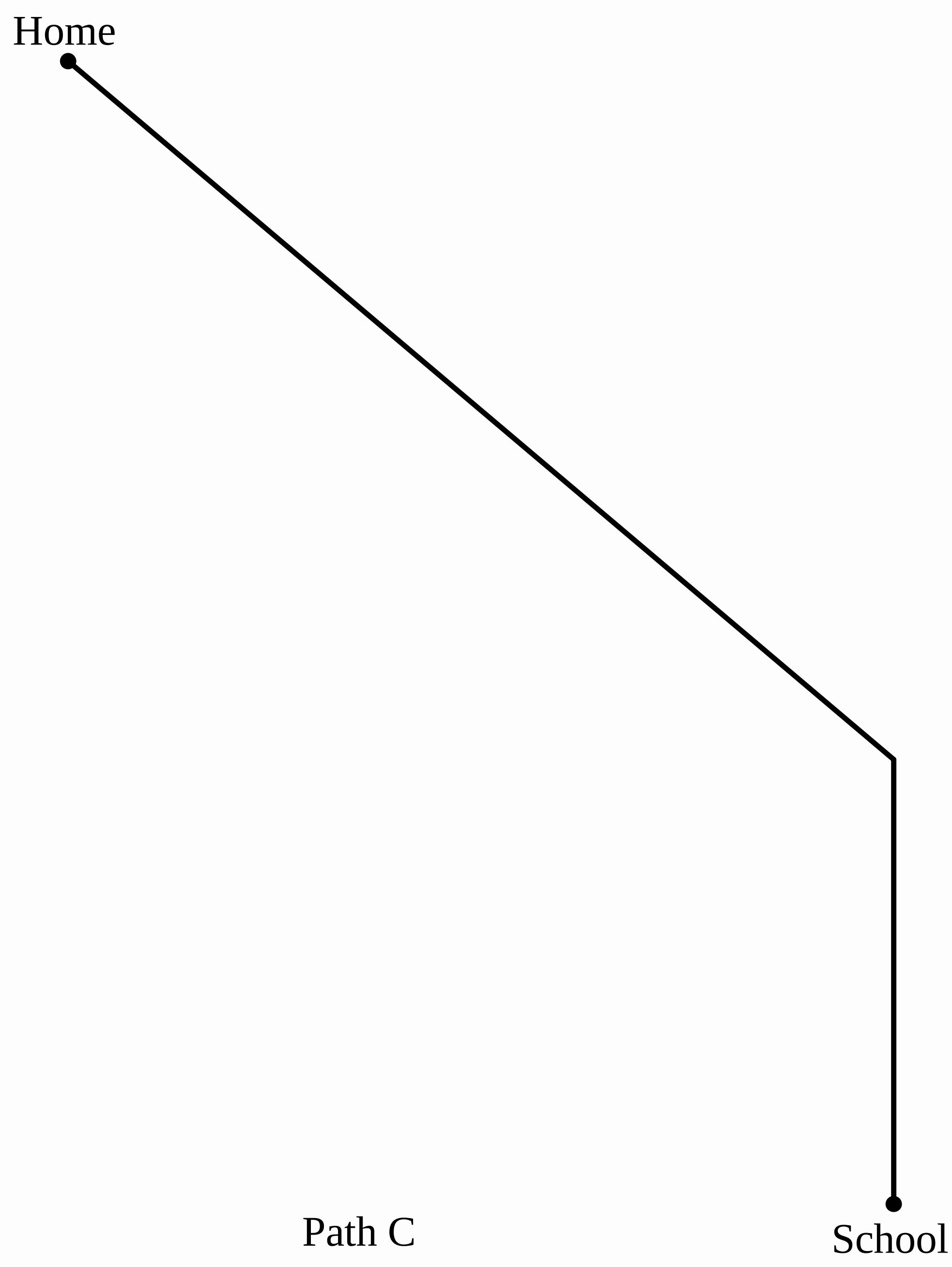
String 2

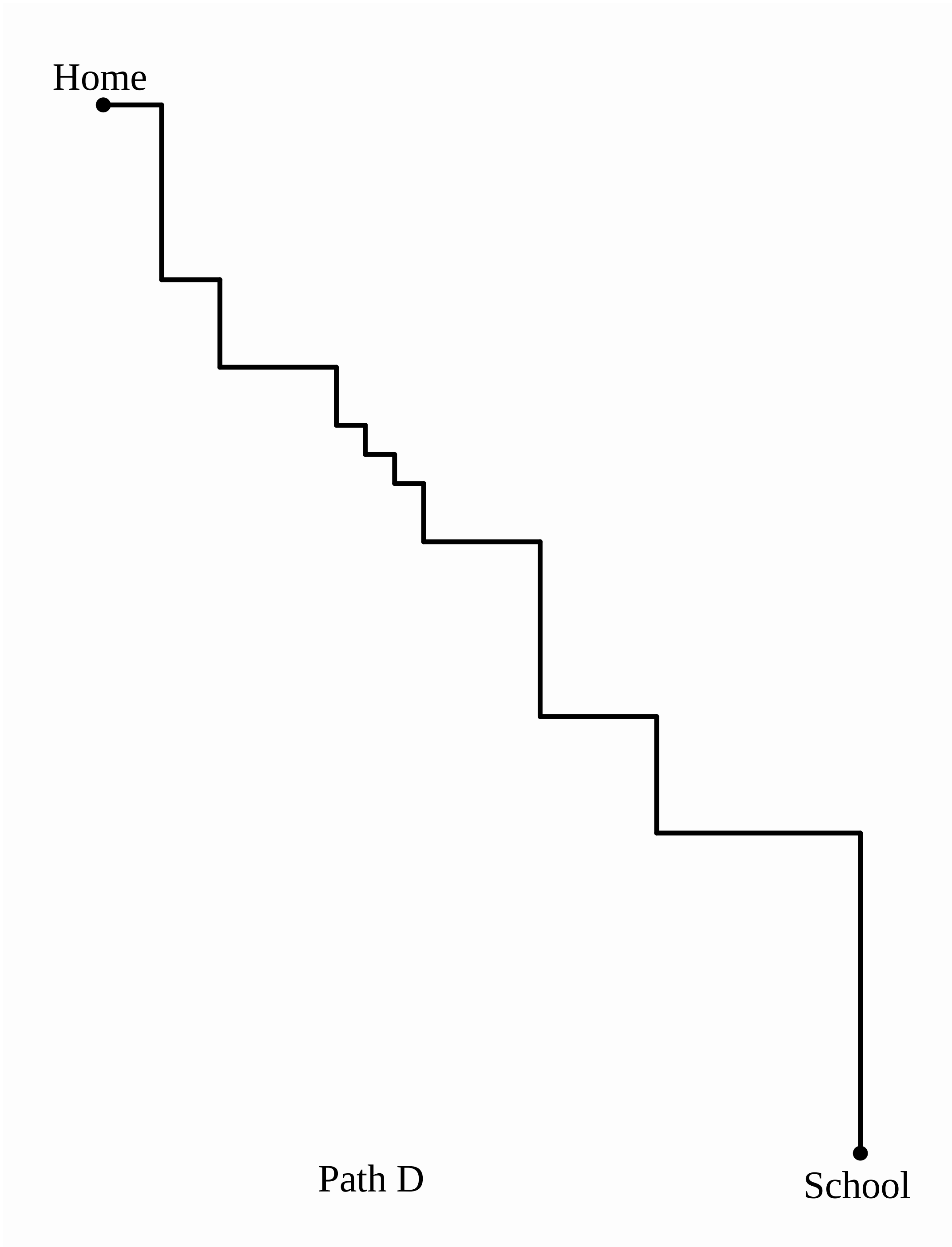


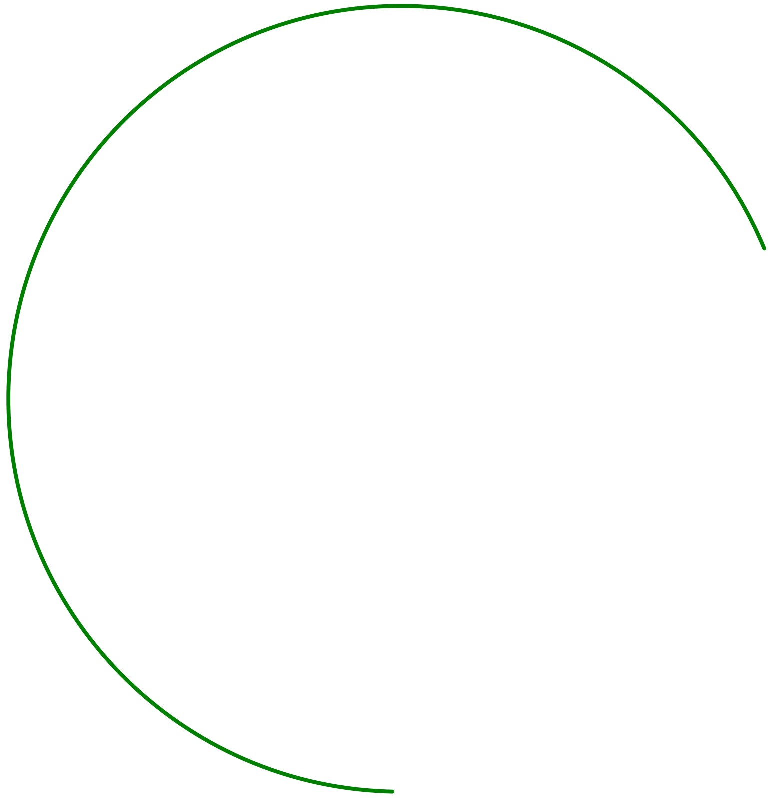
String 3

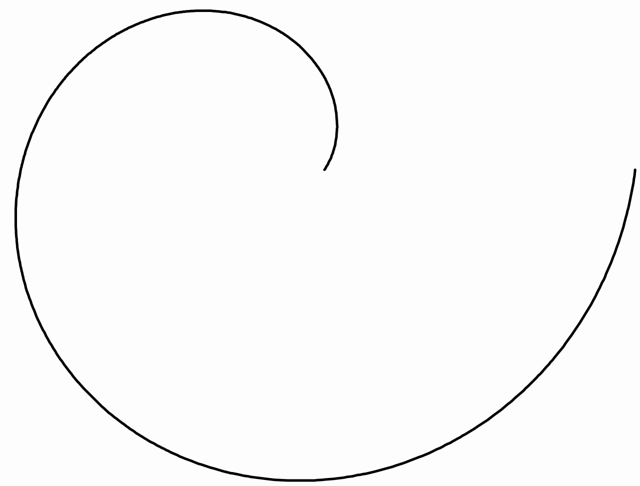

Home
Path A
School

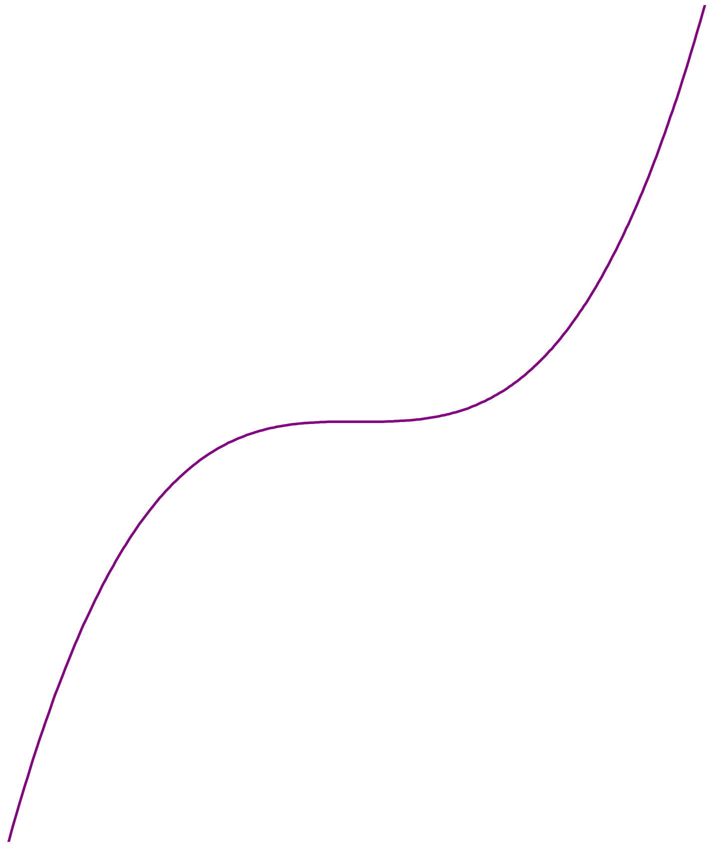

Home
Path B
School

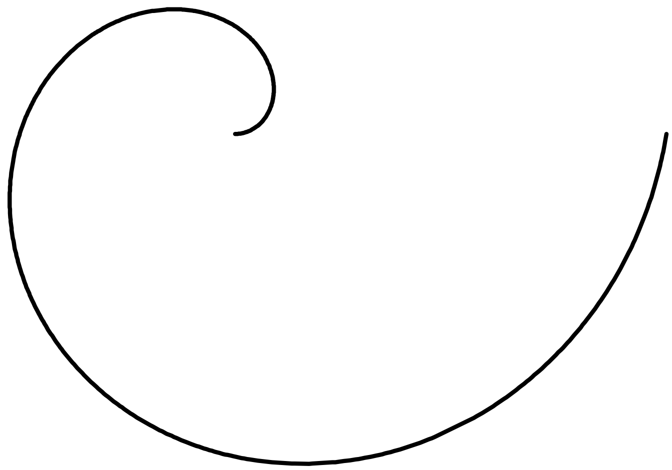

Home
School
Path C

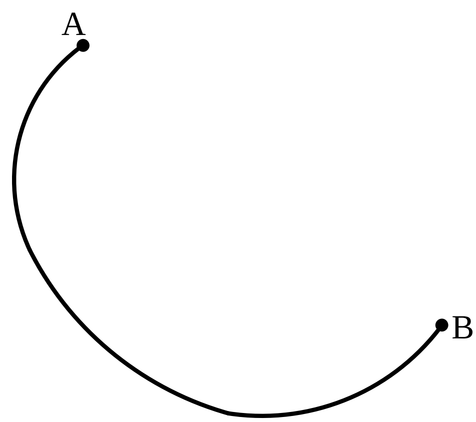

Home
Path D
School



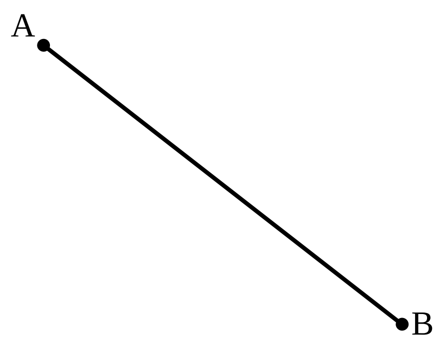




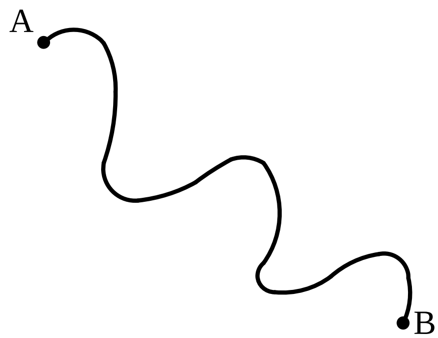




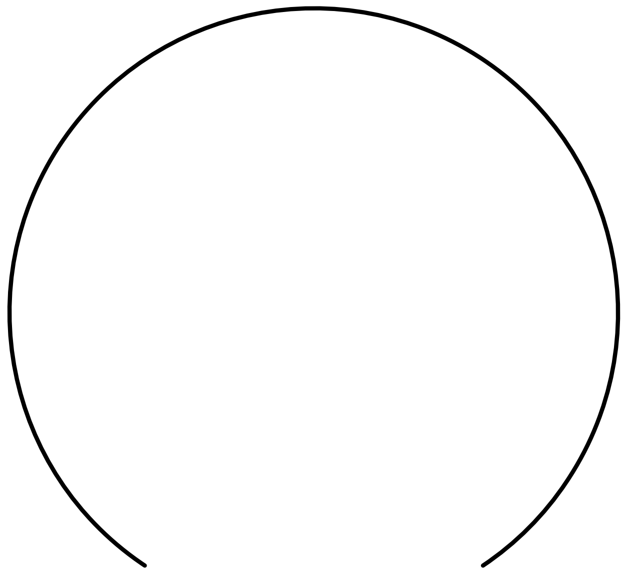
String 1

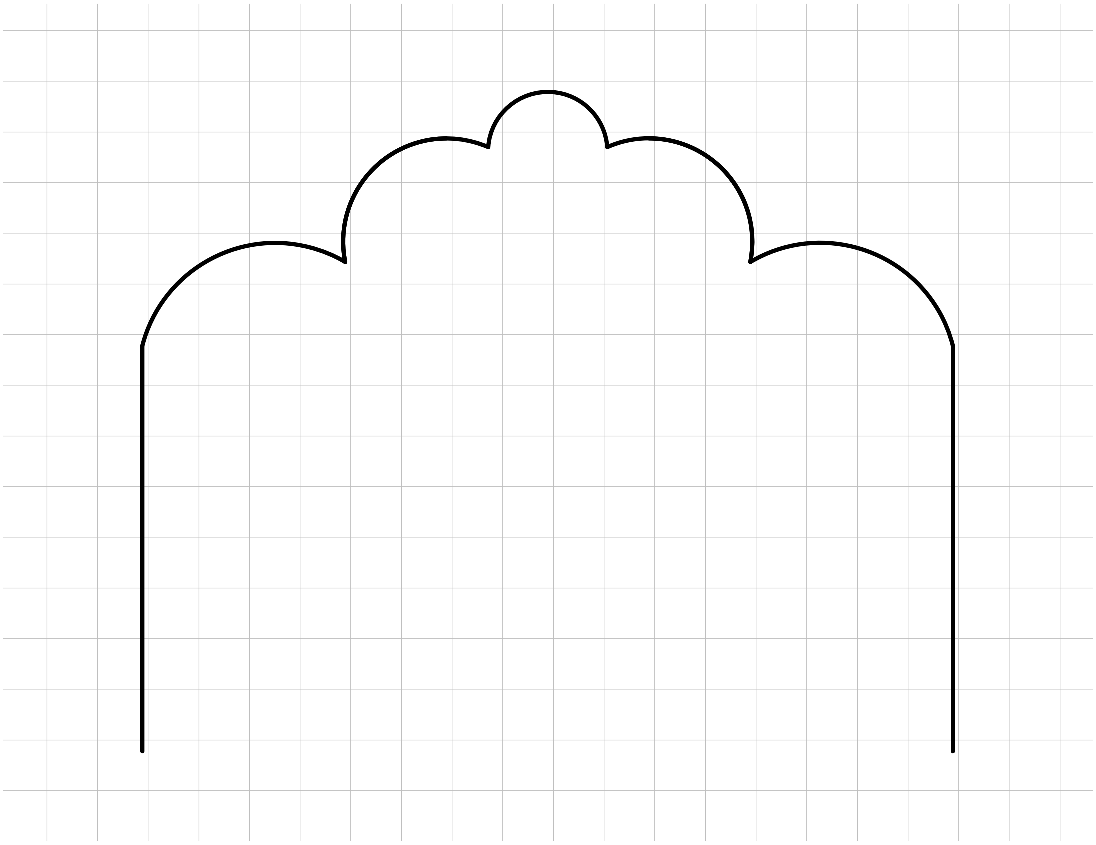


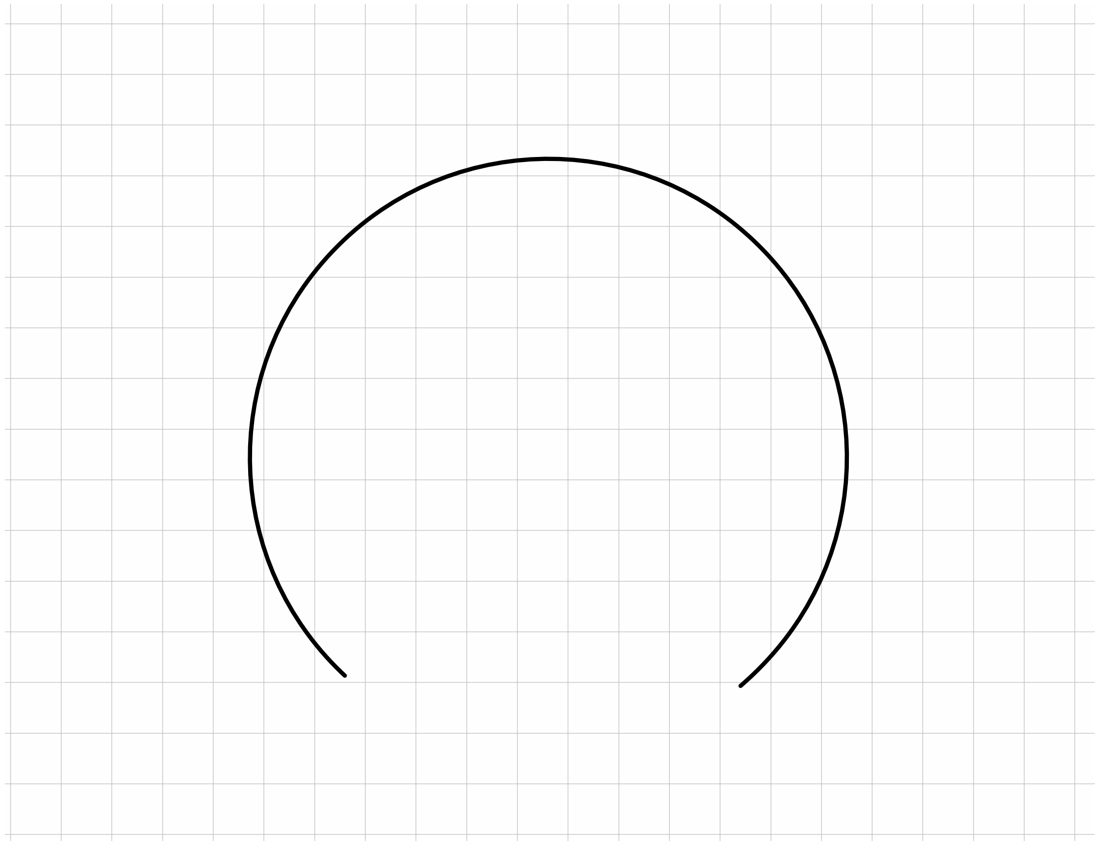
String 2



String 3







APPENDIX D

CODING SCHEME

Code Descriptor	Thematic Category	Corresponding Observable Behaviors: Statements, Gestures, or Manipulations of Tools
Straightness Intuition	Intuition	statement: explained that a path was shortest because it was straight (without providing further justification)
Detour Intuition	Intuition	statement: discussed a path as going out of the way or not being a direct route
Complexity Intuition	Intuition	statement: discussed the number of segments, turns, or angles of a path
Compression Intuition	Intuition	statement: discussed either straightening or bending paths for the purpose of making comparisons
Curve Tightness Intuition	Intuition	statement: discussed a curve as being longer than another because it was curved in more or because it had more curve
Combination of Intuitions	Intuition	statement: used more than one intuition (straightness, detour, complexity, compression, or curve tightness) to defend a single claim

Rejected Intuition	Intuition	statement: rejected conclusion previously defended using an intuitive statement (student may reject an intuition by evoking another intuition, combination of intuitions, or analytical strategy)
Used a Rejected Intuition	Intuition	statement: again used an intuition previously rejected
Indirect Comparison Using Finger Span	Analytical strategy	gesture: placed a finger span across a segment of one path and then placed the same finger span across a segment of another path
Superimposed Pairs of Paths to Directly Compare	Analytical strategy	gesture: placed one path directly on top of another for the purpose of directly comparing by linear extent
Segment Matching Comparison Strategy	Analytical strategy	gesture: matched segments of one path to the segments of another path
Project to Form Right Angle	Analytical strategy	statement: explained that he or she compared (rectilinear paths) by imagining or translating vertical segments horizontally and horizontal segments vertically
Accumulating Length Comparison Strategy	Analytical strategy	gesture: superimposed pairs of paths, rotated one of the paths while accumulating the length of the first along the second
Rate Comparison Strategy	Analytical Strategy	statement: discussed traversing paths or segments of paths at the same rate for the purpose of comparing

Imposed Internal Unit	Analytical Strategy	manipulation of tools: drew approximately evenly spaced hash marks for the purpose of directly comparing
Chord Iteration Strategy	Analytical Strategy: Direct Measurement	manipulation of tools: iterated a stick as a chord on the interior of the curve when comparing a curve to a straight object
Continuous comparison strategy to estimate	Analytical Strategy: Direct Measurement	Gesture or manipulation of tools: moved a finger or a straight object along a path in a continuous motion for the purpose of comparing two or more paths
Tangent Iteration Strategy	Analytical Strategy: Direct Measurement	manipulation of tools: iterated a stick as a tangent on the exterior of the curve when comparing a curve to a straight object
Mixed Unit Iteration Strategy	Analytical Strategy: Direct Measurement	manipulation of tools: iterated a stick sometimes placing it as a chord on the interior of the curve, sometimes as a tangent on the exterior of the curve, and other times placing the stick directly on the curve when comparing a curve to a straight object
Path Intersection Iteration Strategy	Analytical Strategy: Direct Measurement	manipulation of tools: iterated a stick by attempting to place it directly on the curve when comparing a curve to a straight object
Adjusting point of tangency iteration strategy	Analytical Strategy: Direct Measurement	manipulation of tools: placed a stick as a tangent aligned with one end of the curve and rotated the stick, adjusting the point of tangency and accumulating the length of part of the curve along the stick

Modified circumference formula strategy	Analytical Strategy: Indirect Measurement	statement: discussed comparing the stick to the radius of a partial circle-shaped curve, visually estimated the fraction of a circle represented by the curve, and modified and applied the formula for the circumference of a circle accordingly
Used the Whole Stick as a Unit	Analytical Strategy Related to Unit	manipulation of tools: placed the whole stick (as a chord, tangent, or directly on the curve) and used it as a unit to compare curves rather than fracturing and operating on partial stick units
Fractured Non-standard Unit Once at the End of the Curve	Analytical Strategy Related to Unit: Fractured Unit	manipulation of tools: when a full stick unit did not fit along the curve at the end, discussed using a partial stick unit (such as one half or one third of the stick) to measure the last segment of the curve
Fractured Non-standard Unit in the Tightest Part of the Curve	Analytical Strategy Related to Unit: Fractured Unit	manipulation of tools: operated on partial stick units in the tightest part of the curve for the purpose of increasing precision
Fractured Non-standard Unit Around the Entire Curve	Analytical Strategy Related to Unit: Fractured Unit	manipulation of tools: operated on partial stick units around the entire curve for the purpose of increasing precision
Counted Partial Unit as a Whole Unit	Analytical Strategy Related to Unit	statement: when only a partial stick unit would fit at the end of the curve, counted this (as well as the other stick unit segments) as a full stick unit
Compensated for Curvature	Analytical Strategy Related to Unit	statement: after comparing using another analytical strategy related to unit, rounded (or added or subtracted) to this count of units to account for an over- or underestimate due to representing a curve with straight segments

Applied Benchmark	Analytical Strategy Related to Unit: Applied Mental Units	statement: discussed thinking about a previously known measurement when comparing a curve and a straight object
Applied Conceptual Standard Unit	Analytical Strategy Related to Unit: Applied Mental Units	statement: discussed applying mental image of a standard unit (a centimeter or an inch)
Tangent Curved Unit Iteration Strategy	Analytical Strategy with Embedded Intuition: Direct Measurement	manipulation of tools: applied the tangent iteration strategy, but allowed the stick to extend beyond the curve; placed a tick mark to represent the end of this stick unit by imagining where the curve and the stick would meet by mentally curving the stick or mentally straightening part of the curve
Chord Curved Unit Iteration Strategy	Analytical Strategy with Embedded Intuition: Direct Measurement	manipulation of tools: applied the chord iteration strategy, but drew a tick mark before the point of intersection of the stick and the curve; the placement of this tick mark was guided by imagining where the curve and the stick would meet by mentally curving the stick or mentally straightening part of the curve
Acknowledged Underestimate	Reflection on Error	statement: discussed a comparison between a curve and a straight object as involving an underestimate
Acknowledged Overestimate	Reflection on Error	statement: discussed a comparison between a curve and a straight object as involving an overestimate
Claimed Answer was not an Over- or Underestimate	Reflection on Error	statement: discussed a comparison between a curve and a straight object as involving neither an over- nor underestimate

Correct Acknowledgement of an Over- or Underestimate	Reflection on Error	statement and manipulation of tools: discussed a comparison between a curve and a straight object as involving an underestimate after having applied the chord iteration strategy, or discussed a comparison between a curve and a straight object as involving an overestimate after having applied the tangent iteration strategy
Incorrect Acknowledgement of an Over- or Underestimate	Reflection on Error	statement and manipulation of tools: discussed a comparison between a curve and a straight object as involving an underestimate after having applied the tangent iteration strategy, discussed a comparison between a curve and a straight object as involving an overestimate after having applied the chord iteration strategy, or claimed not to have over- or underestimated after having applied the chord or tangent iteration strategy
Acknowledgment of an Over- or Underestimate was Neither Correct nor Incorrect	Reflection on Error	Statement and manipulation of tools: discussed a comparison between a curve and a straight object as being an over- or underestimate, or and being neither an over- nor underestimate, after having applied a direct measurement analytical strategy that was not in conflict with their claim
Attended to Symmetry When Measuring with a Ruler	Analytical Strategy	manipulation of tools: measured only part of each shape, attending to the symmetry of the curve, rather than measuring the entire curve